HW # 8 Due 11/13

$\boxed{9.3}$ Let $H(z)$ be the T.F. of a lowpass digital filter with a passband edge at ω_p, stopband edge at ω_s, passband ripple of δ_p, and stopband ripple of δ_s, as indicated in Fig 9.1. Consider a cascade of two identical filters with T.F. $H(z)$. What are the passband/stopband ripples of the cascade at ω_p and ω_s, respectively? Generalize results to M identical sections

Let $G(z) = H^2(z)$ and $G(e^{j\omega}) = H^2(e^{j\omega})$

$$|G(e^{j\omega})| = |H^2(e^{j\omega})| = |H(e^{j\omega})|^2$$

the passband/stopband ripples of $G(e^{j\omega})$ will be

$$\delta_{p,2} = 1 - (1-\delta_p)^2 \quad \text{and} \quad \delta_{s,2} = (\delta_s)^2$$

For M sections, $\delta_{p,M} = 1 - (1-\delta_p)^M \quad \delta_{s,M} = (\delta_s)^M$

$\boxed{9.17}$ Let $H_a(s)$ be a real-coefficient causal and stable analog T.F. with a magnitude response bounded above by 1. Show that the digital T.F. $G(z)$ obtained by bilinear transformation of $H_a(s)$ is a bounded real function.

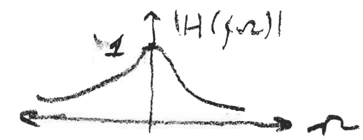
$H_a(s)$ is causal and stable ...

$$G(z) = H_a(s)\Big|_{s = j\frac{2}{T}\tan\left(\frac{\omega}{2}\right)} = H_a\left(j\frac{2}{T}\tan\left(\frac{\omega}{2}\right)\right)$$

$\max(|H_a(j\Omega)|) = 1$

$$|G(e^{j\omega})| = \left|H_a\left(j\frac{2}{T}\tan\left(\frac{\omega}{2}\right)\right)\right|$$

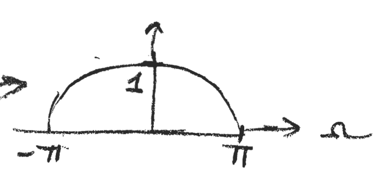
Example: $H_a(s) = \dfrac{\Omega_c}{s + \Omega_c}$



$$G(e^{j\omega}) = \frac{\Omega_c}{j\frac{2}{T}\tan\left(\frac{\omega}{2}\right) + \Omega_c}$$

$$|G(e^{j\omega})| = \frac{\sqrt{\Omega_c^2}}{\sqrt{\frac{4}{T^2}\tan^2\left(\frac{\omega}{2}\right) + \Omega_c^2}}$$

$\max(|G(e^{j\omega})|) = 1 \quad @ \quad (\Omega = \omega = 0)$



$\boxed{9.21}$ Let $H_{LP}(z)$ be an IIR lowpass T.F with a zero/pole at $z = z_k$. Let $H_D(\hat{z})$ denote the bandpass T.F obtained by applying the low-to-bandpass transformation, which moves the zero (pole) at $z = z_k$ of $H_{LP}(z)$ to a new location at $\hat{z} = \hat{z}_k$. Express $\hat{z}_k$ in terms of z_k. If $H_{LP}(z)$ has a zero at $z=-1$, show that $H_D(\hat{z})$ also has a zero at $z = \pm 1$.

Let us use the ~~low~~-to-bandpass transformation (Table 9.1)

$$z^{-1} = -\frac{\hat{z}_k^{-2} - a_1 \hat{z}_k^{-1} + a_2}{a_2 \hat{z}_k^{-2} - a_1 \hat{z}_k^{-1} + 1} \qquad \boxed{\begin{aligned} a_1 &= \frac{2\lambda\rho}{\rho+1} \\ a_2 &= \frac{\rho-1}{\rho+1} \end{aligned}}$$

$$z = -\frac{a_2 \hat{z}_k^{-2} - a_1 \hat{z}_k^{-1} + 1}{\hat{z}_k^{-2} - a_1 \hat{z}_k^{-1} + a_2} = -\frac{a_2 - a_1 \hat{z}_k + \hat{z}_k^2}{1 - a_1 \hat{z}_k + a_2 \hat{z}_k^2}$$

A zero/pole located at $(z - z_k)$ will be mapped to:

$$-\frac{a_2 - a_1 \hat{z}_k + \hat{z}_k^2}{1 - a_1 \hat{z}_k + a_2 \hat{z}_k^2} - z_k = 0 \qquad -a_2 + a_1 \hat{z}_k - \hat{z}_k^2 - z_k + a_1 z_k \hat{z}_k - a_2 z_k \hat{z}_k^2 = 0$$

$$(-1 - a_2 z_k)\hat{z}_k^2 + a_1(1 + z_k)\hat{z}_k + (-a_2 - z_k) = 0$$

$$\underset{a}{\hat{z}_k^2} - \underset{b}{\frac{a_1(1+z_k)}{1 + a_2 z_k}}\hat{z}_k + \underset{c}{\frac{(a_2 + z_k)}{(1 + a_2 z_k)}} = 0$$

$$\hat{z}_k = \frac{\dfrac{a_1(1+z_k)}{(1+a_2 z_k)} \pm \sqrt{\left[\dfrac{a_1(1+z_k)}{(1+a_2 z_k)}\right]^2 - \dfrac{4(a_2 + z_k)}{(1 + a_2 z_k)}}}{2}$$

$$= \frac{a_1(1+z_k)}{2(1+a_2 z_k)} \pm \sqrt{\left[\frac{a_1(1+z_k)}{2(1+a_2 z_k)}\right]^2 - \frac{(a_2 + z_k)}{1 + a_2 z_k}}$$

Let us find out where the zero/pole at $z=-1$ maps to:

$$\hat{z}_k = \pm\sqrt{\frac{1 - a_2}{1 - a_2}} = \underline{\underline{\pm 1}}$$

$\boxed{9.26}$ Design a bandpass filter with center frequency at $\hat{\omega}_0 = 0.5\pi$ by transforming the bandpass in Eq. 7.79 using lowpass-to-lowpass.

$$H_{BP}(z) = 0.136728736\left(\frac{1 - z^{-2}}{1 - 0.53353098z^{-1} + 0.726542528z^{-2}}\right)$$

Plotting $H_{PB}(z)$ we know that $\omega_0 = 0.4\pi$
So, using low-to-low transformation (Table 9.1)

$$\lambda = \frac{\sin(-0.05\pi)}{\sin(0.45\pi)} = -0.1584$$

$$G_{BP}(z) = H_{BP}(z)\Big|_{z^{-1} = \frac{\hat{z}^{-1} - \lambda}{1 - \lambda\hat{z}^{-1}}} = \frac{a\left[1 - \frac{(\hat{z}^{-1} - \lambda)^2}{(1 - \lambda\hat{z}^{-1})^2}\right]}{1 - b\frac{(\hat{z}^{-1} - \lambda)}{(1 - \lambda\hat{z}^{-1})} + c\frac{(\hat{z}^{-1} - \lambda)^2}{(1 - \lambda\hat{z}^{-1})^2}}$$

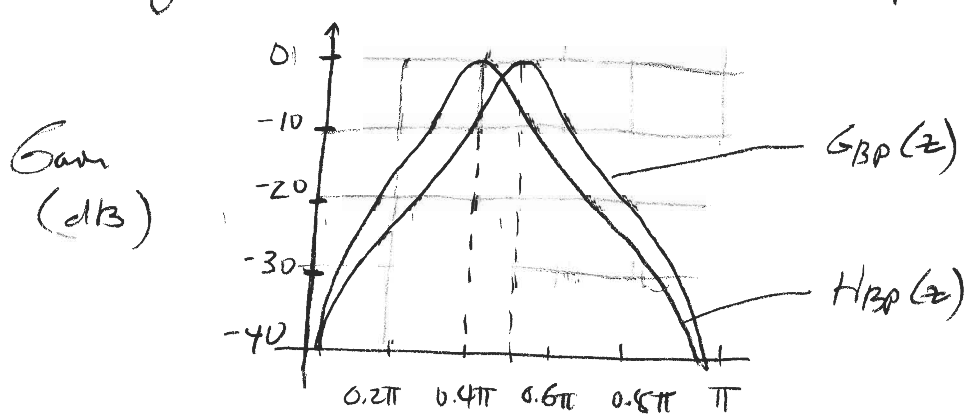
$$= \frac{a\left[(1 - \lambda\hat{z}^{-1})^2 - (\hat{z}^{-1} - \lambda)^2\right]}{(1 - \lambda\hat{z}^{-1})^2 - b(\hat{z}^{-1} - \lambda)(1 - \lambda\hat{z}^{-1}) + c(\hat{z}^{-1} - \lambda)^2}$$

$$= \frac{a\left[(1 - 2\cancel{\lambda\hat{z}^{-1}} + \lambda^2\hat{z}^{-2}) - (\hat{z}^{-2} - 2\cancel{\lambda\hat{z}^{-1}} + \lambda^2)\right]}{\cancel{1} - 2\lambda\hat{z}^{-1} + \lambda^2\hat{z}^{-2} - b(\hat{z}^{-1} - \cancel{\lambda} - \lambda\hat{z}^{-2} + \cancel{\lambda^2\hat{z}^{-1}}) + c(\hat{z}^{-2} - 2\lambda\hat{z}^{-1} + \cancel{\lambda^2})}$$

$$= \frac{a\left[(1 - \lambda^2) + (\lambda^2 - 1)\hat{z}^{-2}\right]}{(1 + b\lambda + c\lambda^2) + (-2\lambda - b - b\lambda^2 - 2c\lambda)\hat{z}^{-1} + (\lambda^2 + b\lambda + c)\hat{z}^{-2}}$$

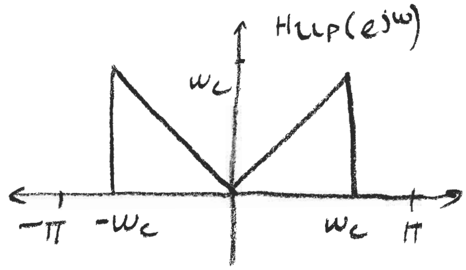
$$= \frac{0.1333 - 0.1333\hat{z}^{-2}}{0.9337 + 0.66712\hat{z}^{-2}}$$

Using MATLAB to plot gains of both filters.



HW #9 . Due 11/25

10.10 Determine the impulse response $h_{LLP}[n]$ of a zero-phase ideal linear passband lowpass filter characterized by a frequency response.



$$H_{LLP}(e^{j\omega}) = \begin{cases} |\omega| & |\omega| \le \omega_c \\ 0 & \text{O.W} \end{cases}$$

$$h_{LLP}[n] = \frac{1}{2\pi}\left[-\int_{-\omega_c}^{0} \omega e^{j\omega n}\,d\omega + \int_{0}^{\omega_c} \omega e^{j\omega n}\,d\omega\right] = \frac{1}{2\pi}\left[-\left(\frac{\omega e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2}\right)\Bigg|_{-\omega_c}^{0} + \right.$$

$$\left.\left(\frac{\omega e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2}\right)\Bigg|_{0}^{\omega_c}\right] = \frac{1}{2\pi}\left[-\frac{1}{n^2} - \frac{\omega_c e^{-j\omega_c n}}{jn} + \frac{e^{-j\omega_c n}}{n^2} + \right.$$

$$\left.\frac{\omega_c e^{j\omega_c n}}{jn} + \frac{e^{j\omega_c n}}{n^2} - \frac{1}{n^2}\right] = \frac{1}{2\pi}\left[\frac{\omega_c e^{j\omega_c n} - \omega_c e^{-j\omega_c n}}{jn} + \frac{e^{j\omega_c n} + e^{-j\omega_c n}}{n^2} - \frac{2}{n^2}\right]$$

$$= \frac{\omega_c}{\pi n}\sin(\omega_c n) + \frac{\cos(\omega_c n)}{\pi n^2} - \frac{1}{\pi n}$$

10.15 For each of the lowpass filter specifications given below, design an FIR filter with the smallest length meeting the specifications using the window-based approach, and plot its magnitude response using MATLAB.

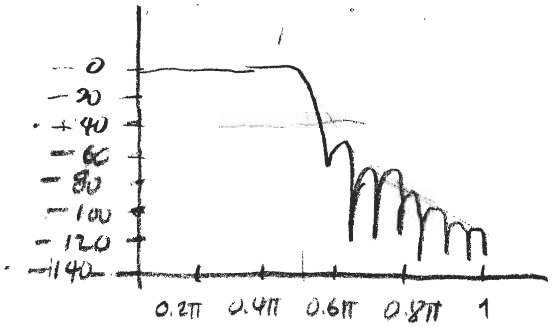
a) $\omega_p = 0.47\pi$, $\omega_s = 0.59\pi$, $\delta_p = 0.001$, $\delta_s = 0.007$

$\alpha_s = -20\log_{10}(0.007) = 43.098$ dB

From Table 10.2, we can achieve minimum stopband attenuation using Hann, Hamming or Blackman window.

Hann will have the lowest filter length

$0.12\pi = \frac{3.11\pi}{M}$ $M = 25.917$ $N = \lceil 2M+1 \rceil = 53$

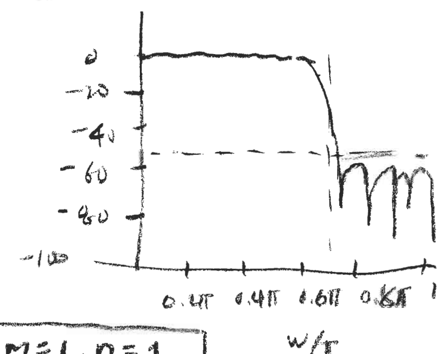


(b) $\omega_p = 0.61\pi$, $\omega_s = 0.78\pi$, $\delta_p = 0.001$, $\delta_s = 0.002$,

$\alpha_s = -20\log_{10}(0.002) = 53.979$ dB

From Table 10.2 we see that Hamming or Blackman can provide ~~this~~ attenuation.

$$M = \frac{3.32\pi}{0.17\pi} = 19.529 \quad N_{Hamm} = \lceil 2M+1 \rceil = 41$$



$\boxed{\begin{array}{l} L=3, M=1, n=1 \\ 10.29 \quad L=4, M=2, n=1 \end{array}}$

In problem 9.3, we considered filtering by a cascade of a number of identical filters. In the case of an FIR filter $H(z)$ with a symmetric impulse response, improved passband/stopband performance can be achieved by employing the FILTER SHARPENING APPROACH in which $G(z)$ is implemented as

$$G(z) = \sum_{\ell=1}^{L} \alpha_\ell [H(z)]^\ell \qquad \{\alpha_\ell\} \text{ are real coeffs}$$

It follows that $G(z)$ is also an FIR filter with a symmetric impulse response. Let x denote a specific amplitude response of $H(z)$ at freq. ω. If we denote the value of $G(z)$ at this value of ω as $P(x)$, then

$$P(x) = \sum_{\ell=1}^{L} \alpha_\ell x^\ell$$

For BR T.F. $H(z)$, $0 \le x \le 1$, where $x=0$ is in stopband and $x=1$ is in passband. If we desire $P(0)=0$ & $P(1)=1$ and constraining the slope at $x=0$ & $x=1$.

To improve the performance of $G(z)$ in stop band and pass bands.

$$\left.\frac{d^k P(x)}{dx^k}\right|_{x=0} = 0, \quad k = 1, 2, \ldots, n$$

$$\left.\frac{d^k P(x)}{dx^k}\right|_{x=1} = 0 \quad k = 1, 2, \ldots m$$

where $m + n = L - 1$

a) Determine $\{\alpha_\ell\}$ for $L = 3$

$L = 3 = m + n + 1$ then pick $m = n = 1$

$P(x) = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$

✓ $P(0) = 0$ satisfied

✓ $P(1) = \alpha_1 + \alpha_2 + \alpha_3 = 1$

✓ $\left.\frac{d\,P(x)}{dx}\right|_{x=0} = \alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 \Big|_{x=0} = \alpha_1 = 0$

✓ $\left.\frac{d\,P(x)}{dx}\right|_{x=1} = \alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 \Big|_{x=1} = \alpha_1 + 2\alpha_2 + 3\alpha_3 = 0$

Solving these eqns

$\alpha_1 = 0$ $\qquad \alpha_2 = 1 - \alpha_3 = 3$

$\alpha_2 + \alpha_3 = 1$ $\qquad 2 - 2\alpha_3 + 3\alpha_3 = 0$

$2\alpha_2 + 3\alpha_3 = 0$ $\qquad \alpha_3 = -2$

$P(x) = 3x^2 - 2x^3$

b) Determine $\{\alpha_\ell\}$ for $L = 4$

$L = 4 = m + n + 1$ choose $m = 2$ & $n = 1$

$p(x) = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4$

✓ $P(0) = 0$

✓ $p(1) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$

✓ $\left.\frac{dP(x)}{dx}\right|_{x=0} = \alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 + 4\alpha_4 x^3 \Big|_{x=0} = 0 \Rightarrow \alpha_1 = 0$

✓ $\left.\frac{d^2 P(x)}{dx^2}\right|_{x=0} = 2\alpha_2 + 6\alpha_3 x + 12\alpha_4 x^2 \Big|_{x=0} = 0 \Rightarrow 2\alpha_2 = 0 \quad \alpha_2 = 0$

✓ $\left.\frac{d\,P(x)}{dx}\right|_{x=1} = 0 \Rightarrow \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 = 0$

Determin $\{\alpha_\ell\}$ for $l=4$

$L = 4 = m+1+1 \qquad m=1 \ \& \ n=2$

$P(x) = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4$

$P(0) = 0$

$P(1) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$

$$\left.\frac{dP(x)}{dx}\right|_{x=0} = \alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 + 4\alpha_4 x^3 \Big|_{x=0} \implies \alpha_1 = 0$$

$$\left.\frac{dP(x)}{dx}\right|_{x=1} = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 = 0$$

$$\left.\frac{d^2P(x)}{dx^2}\right|_{x=1} = 2\alpha_2 + 6\alpha_3 x + 12\alpha_4 x^2 \Big|_{x=1} = 0$$

$$2\alpha_2 + 6\alpha_3 + 12\alpha_4 = 0$$

$\alpha_2 + \alpha_3 + \alpha_4 = 1 \qquad \alpha_1 = 0$

$2\alpha_2 + 3\alpha_3 + 4\alpha_4 = 0 \qquad \alpha_2 = 6$

$2\alpha_2 + 6\alpha_3 + 12\alpha_4 = 0 \qquad \alpha_3 = -8$

$\alpha_4 = 3$

$$\underline{P(x) = 6x^2 - 8x^3 + 3x^4}$$

$\alpha_3 + \alpha_4 = 1$ $\qquad \alpha_3 = 1 - \alpha_4 = 4$

$3\alpha_3 + 4\alpha_4 = 0$ $\qquad 3 - 3\alpha_4 + 4\alpha_4 = 0$

$\alpha_4 = -3$

$$P(x) = 4x^3 - 3x^4$$

$\boxed{\text{M10.22}}$ The m-file FIR2 is employed to design FIR filters with arbitrarily shaped magnitude responses. Using this function, design a FIR of order 70 with 3 different magnitude levels:

$0.2 \rightarrow 0 - 0.35\pi$

$1.0 \rightarrow 0.4\pi - 0.7\pi$

$0.6 \rightarrow 0.72\pi - \pi$

Plot gain response of filter.

```
Fr =[0 0.35 0.4 0.7 0.72 1];
Mag=[0.2 0.2 1 1 0.6 0.6];
N=70
b=fir2(N, Fr,Mag);
[H,w]= freqz(b,1,512);
plot(w/pi, abs(H)); xlabel('w'); ylabel('Mag');
axis([0 1 0 1.2]);
```