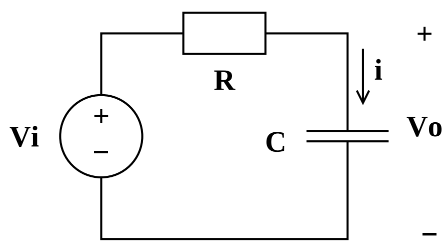
EE 211 Circuit and Signals I, Fall 2012
Exam 3 November 19, 2012
Solution

Rules: This is a closed-book exam. You may use only your brain, a calculator and pen/paper. Each numbered question counts equally toward your grade.

Grading policy: Only boxed answers count. If you have boxed the correct answer and show enough relevant math you get 10. If only one of them you get 5. Otherwise you get 0. Always include the unit on every answer and reduce to simple form!



Assume $R = 1\,\Omega$ and $C = 1\,\mathrm{F}$

1. **Derive the differential equation for $v_O(t)$**

   Using KVL

$$v_I = iR + v_O = CR\frac{dv_O}{dt} + v_O$$

2. **If $v_I(t) = 1\,\mathrm{V} \times u(t)$ find $v_O(t)$.**

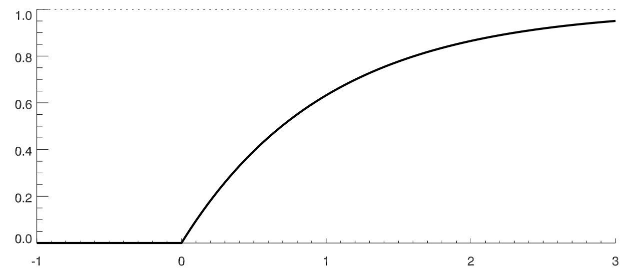
   The general solution is

$$v_O = \left[Ae^{-\frac{t}{RC}} + B\right] u(t) \tag{1}$$

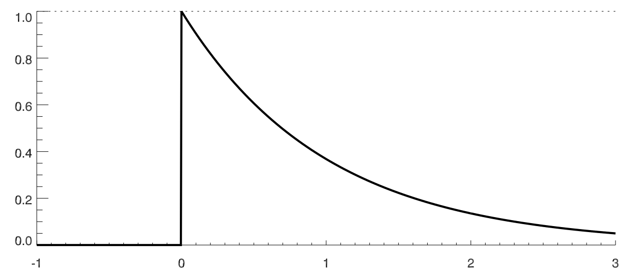
   The final values is $v_o \to 1\,\mathrm{V}$ for $t \to \infty$, so $C = 1\,\mathrm{V}$. The initial value is $v_o(0) = 0\,\mathrm{V}$, so $A = -1\,\mathrm{V}$. Final solution is then

$$\boxed{v_O = \left[1\,\mathrm{V} - e^{-\frac{t}{1\,\mathrm{s}}}\right] u(t)}$$

3. **Plot v_O from previous question carefully (that means including labelling axis and voltage levels).**



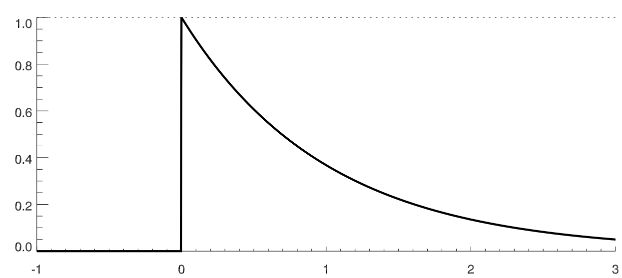
4. For the same $v_I(t)$ sketch $i(t)$.



5. If $v_I(t) = 1\,\mathrm{V}\cdot\mathrm{s} \times \delta(t)$ find $v_O(t)$ and plot it carefully.

This will be a step function in $v_O(t)$ followed by a decay to zero. Identical to $i(t)$ from before.

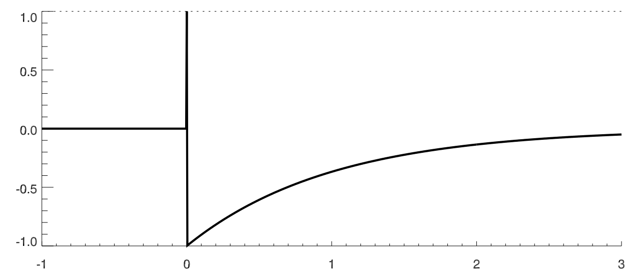
$$\boxed{v_o(t) = \left[1\,\mathrm{V} e^{-\frac{t}{1\,\mathrm{s}}}\right] u(t)}$$



6. For the same $v_I(t)$ find $i(t)$ and plot it carefully.

This will be a positive current spike to infinity followed by a negative current starting at $-1\,\mathrm{A}$ decaying to zero.

$$\boxed{i(t) = 1\,\mathrm{A} \times \delta(t) - \left[e^{-\frac{t}{1\,\mathrm{s}}}\right] u(t)}$$



7. If $v_I(t) = 1\,\mathrm{V} \cdot \cos(2\,\mathrm{s}^{-1} \times t)$, $R = 1\,\Omega$, $C = 2\,\mathrm{F}$, what is the amplitude of $v_O(t)$, and what is its phase relative to $v_I(t)$?

Use phasors for this

$$v_O = \frac{Z_C}{Z_C + Z_R} v_I = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC} v_I$$

Amplitude is

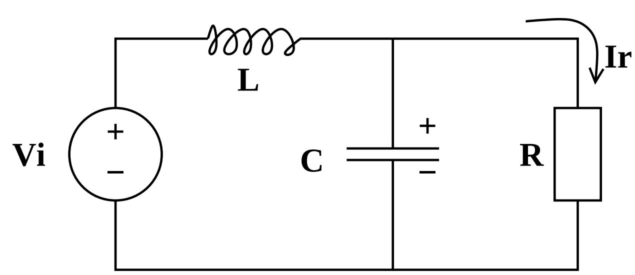
$$|V_O| = \frac{1}{\sqrt{1 + \omega RC}} |V_I| = \frac{1}{\sqrt{1 + 2 \times 2}} = \frac{1}{\sqrt{5}}$$

$$\boxed{|V_O| = 0.447 |V_I|}$$

whereas the phase is

$$\angle V_O - \angle V_I = -\tan^{-1} \omega RC = -\tan^{-1} 4$$

$$\boxed{\angle V_O - \angle V_I = -76^\circ}$$



8. Derive the differential equation for $i_R(t)$ in the standard form.

$$i_R = i_L - i_C = i_L - C\frac{dv_C}{dt} \qquad \frac{di_R}{dt} = \frac{di_L}{dt} - C\frac{d^2 v_C}{dt^2}$$

$$\frac{di_R}{dt} = \frac{v_L}{L} - C\frac{d^2 v_C}{dt^2} \qquad \frac{di_R}{dt} = \frac{v_I}{L} - \frac{v_C}{L} - C\frac{d^2 v_C}{dt^2}$$

$$\frac{di_R}{dt} = \frac{v_I}{L} - \frac{R}{L}i - RC\frac{d^2 i_R}{dt^2} \qquad \frac{1}{RC}\frac{di_R}{dt} = \frac{1}{RLC}v_I - \frac{1}{LC}i - \frac{d^2 i_R}{dt^2}$$

$$\boxed{\frac{d^2 i_R}{dt^2} + \frac{1}{RC}\frac{di_R}{dt} + \frac{1}{LC}i = \frac{1}{RLC}v_I}$$

9. If $C = 1\,\mathrm{F}$ and $L = 1\,\mathrm{H}$ pick R to get $\omega_d = 2\,\mathrm{s}^{-1}$.

$$\omega_n^2 = \frac{1}{LC} = 1\,\mathrm{s}^{-2} \qquad \omega_d = \sqrt{\alpha^2 - \omega_n^2}$$

$$\alpha = \sqrt{\omega_d^2 + \omega_n^2} = \sqrt{2^2 + 1^2} = \sqrt{5} = 2.24\,\mathrm{s}^{-1}$$

$$R = \frac{1}{2\alpha C} = \frac{1}{2 \times 2.24 \times 1}$$

$$\boxed{R = 0.223\,\Omega}$$

10. For $v_I(t) = 1\,\mathrm{V} \times u(t)$ derive the initial conditions for $i_R(t)$.

The initial current is zero because the inductor blocks current, $i_R(0) = 0$. The initial slope of the current is also zero because

$$\frac{di_R}{dt} = \frac{1}{R}\frac{dv_C}{dt} = \frac{1}{R}\frac{i_C}{C}$$

since $i_C(0) = 0$ we get

$$\boxed{i_R(0) = 0 \qquad i_R'(0) = 0}$$

11. Sketch $i_R(t)$ taking care to get the initial value, final value, and initial slope correct.

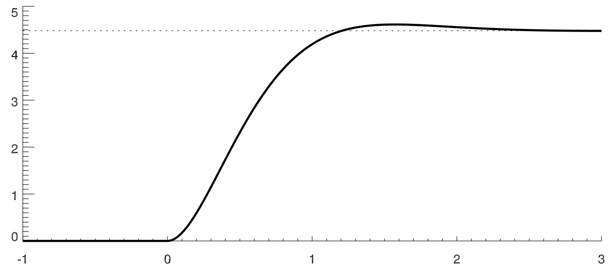
The equation must look something like this

$$i_R(t) = e^{-\alpha t}\left[A\cos(\omega_d t) + B\sin(\omega_d t)\right] + C$$

where $C = \frac{1\,\mathrm{V}}{R} = 4.48\,\mathrm{A}$, and $A = -C$. To get the derivative to be zero we do

$$-\alpha A + \omega_d B = 0 \qquad B = \frac{\alpha A}{\omega_d} = \frac{2.24 \times 4.48}{2} = 5.02\,\mathrm{A}$$

$$\boxed{i_R(t) = e^{-2.24\,\mathrm{s}^{-1}t}\left[-4.48\cos\,2\,\mathrm{s}^{-1}t + 5.02\sin\,2\,\mathrm{s}^{-1}t\right] + 4.48\,\mathrm{A}}$$



12. Sketch $v_C(t)$ for the same case.

$v_C(t)$ looks identical to $i_R(t)$, with R being the constant of proportionality.