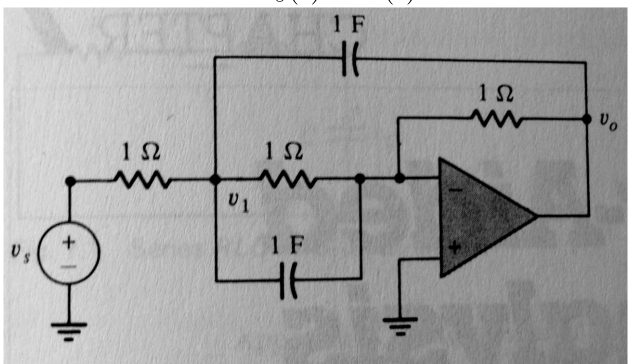
EE 211 Circuit and Signals I, Fall 2012
Example October 31, 2012

Problem 6.34. For the op-amp circuit shown in Fig. P6.34, find the zero-state step response $v_1(t)$ for the case that $v_s(t) = u(t)$ V.



First, come up with the general differential equation for v_1 for $t > 0$.

Nodal analysis at v_1 produces

$$\frac{v_s - v_1}{R} + C\frac{d\left(v_o - v_1\right)}{dt} - \frac{v_1}{R} - C\frac{dv_1}{dt} = 0 \tag{1}$$

we need to eliminate v_o from that equation. We can, for example, say that

$$v_o = -i_F R$$

where i_F is the current flowing from inveting input to v_o, and then

$$i_F = \frac{v_1}{R} + C\frac{dv_1}{dt}$$

Inserting in equation 1 we get

$$\frac{v_s - v_1}{R} - C\frac{d}{dt}\left[v_1 + RC\frac{dv_1}{dt}\right] - C\frac{dv_1}{dt} - \frac{v_1}{R} - C\frac{dv_1}{dt} = 0$$

$$\frac{v_s}{R} - \frac{v_1}{R} - C\frac{dv_1}{dt} - RC^2\frac{d^2v_1}{dt^2} - C\frac{dv_1}{dt} - \frac{v_1}{R} - C\frac{dv_1}{dt} = 0$$

Assembling same derivatives of v_1 we get

$$RC^2\frac{d^2v_1}{dt^2} + 3C\frac{dv_1}{dt} + \frac{2}{R}v_1 = \frac{v_s}{R}$$

$$\frac{d^2v_1}{dt^2} + \frac{3}{RC}\frac{dv_1}{dt}xs + \frac{2}{R^2C^2}v_1 = \frac{v_s}{R^2C^2} \tag{2}$$

Now we have it in the standard form and

$$\alpha = \frac{1}{2}\frac{3}{RC} = \frac{3}{2RC} \qquad \omega_n = \sqrt{\frac{2}{R^2C^2}} = \frac{\sqrt{2}}{RC}$$

What mode is this?

$$\alpha = \frac{3}{2RC} = \frac{3}{2} \qquad \omega_n = \frac{\sqrt{2}}{RC} = \sqrt{2} \approx 1.41$$

Since $\alpha > \omega_n$ the mode is overdamped with the solution

$$v_1 = A_1 e^{s_1 t} + A_2 e^{s_2 t} + K$$

where

$$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_n^2} = -\frac{3}{2} \pm \sqrt{\frac{9}{4} - 2} = -\frac{3}{2} \pm \frac{1}{2} = -1, -2$$

For $t \to \infty$ the capacitors act as open circuits and $v_1 = \frac{v_s}{2} = K$. The solution is now

$$v_1 = A_1 e^{-1t} + A_2 e^{-2t} + \frac{v_s}{2} \tag{3}$$

Next we need to use the initial conditions. We are looking for the zero-state response, so we know that for $t < 0$ $v_1(t) = 0$ and $v_1'(t) = 0$.

For $t > 0$ we have that

$$v_1(t) = v_C(t) = \frac{1}{C} \int_0^t i_C \, dt$$

Since the current is finite and the integral is over zero time it must be that $v_1(0+) = 0$. and thus

$$A_1 + A_2 + \frac{v_s}{2} = 0 \tag{4}$$

Next we need to find the initial condition on $v_1'(0+)$. Notice $v_0 = v_1 = 0$ because the voltage across the capacitor is initially zero. Since we have from earlier

$$v_o = -i_F R = -v_1 + RC \frac{dv_1}{dt}$$

we get

$$\frac{dv_1}{dt}(0) = 0 \tag{5}$$

which gives us

$$-A_1 - 2A_2 = 0 \tag{6}$$

From this we get that $A_2 = -\frac{A_1}{2}$. Inserting into equation 4 we get

$$A_1 - \frac{A_1}{2} + \frac{v_s}{2} = 0$$

$$A_1 = -v_s = -1 \tag{7}$$

and then

$$A_2 = -\frac{A_1}{2} = \frac{v_s}{2} = \frac{1}{2} \tag{8}$$

The expression for v_1 now looks like

$$\begin{aligned} v_1(t) &= \begin{cases} 0 & t < 0 \\ -e^{-t} + \frac{1}{2}e^{-2t} + \frac{1}{2} & 0 \le t \end{cases} \\ &= \left[-e^{-t} + \frac{1}{2}e^{-2t} + \frac{1}{2}\right] u(t) \end{aligned} \tag{9}$$

and here is a plot of that curve and its components.