EE 322 Advanced Analog Electronics, Spring 2012
Homework #11 solution

SS 12.9. A third-order low-pass filter has transmission zeros at $\omega = 2\,\text{rad/s}$ and $\omega = \infty$. Its natural modes are at $s = -1$ and $s = -0.5 \pm j0.8$. The DC gain is unity. Find $T(s)$.

We construct the transfer function by multiplying terms $s - z$ in the numerator (except for zeros at infinity), and terms $s - p$ in the denominator. Thus we get

$$T(s) = \frac{s-2}{(s+1)\,(s+0.5-j0.8)\,(s+0.5+j0.8)}$$

SS 12.13. Calculate the value of attenuation obtained at a frequency 1.6 times the 3-dB frequency of a seventh-order Butterworth filter, and compare it to the first order filter.

The 3-dB frequency of the low-pass filter is the frequency where the square amplitude of the transmission function drops to half of its DC value,

$$\frac{1}{2} = \frac{1}{1+\epsilon^2\left(\frac{\omega_{3\text{dB}}}{\omega_c}\right)^{2N}}$$

First, let's find the 3 dB frequency,

$$\epsilon^2\left(\frac{\omega_{3\text{dB}}}{\omega_c}\right)^{2N} = 1$$

$$\omega_{3\text{dB}} = \omega_c\left(\frac{1}{\epsilon^2}\right)^{\frac{1}{2N}} = \frac{\omega_c}{\sqrt[N]{\epsilon}}$$

Now inserting *omega* $= 1.6\omega_{3\text{dB}}$ into the expression we get

$$|T\left(j\omega = j1.6\times\omega_{3\text{dB}}\right)| = \frac{1}{\sqrt{1+\epsilon^2\left(\frac{1.6\times\omega_c}{\sqrt[n]{\epsilon}\omega_c}\right)^{2N}}} = \frac{1}{\sqrt{1+1.6^{2N}}}$$

For $N = 7$ we get

$$T = \frac{1}{\sqrt{1+1.6^{14}}} = 0.037$$

For a first order filter we know that the transmission function is

$$T(j\omega) = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_{3\text{dB}}}\right)^2}}$$

Inserting $\omega/\omega_{3\text{dB}} = 1.6$ we get

$$T(j1.6\omega_{3\text{dB}}) = \frac{1}{\sqrt{1+1.6^2}} = 0.53$$

SS 12.15. Design a Butterworth filter that meets the following low-pass specifications: $f_p = 10\,\text{kHz}$, $A_{\max} = 2\,\text{dB}$, $f_s = 15\,\text{kHz}$, and $A_{\min} = 15\,\text{dB}$. Find N, the natural modes, and $T(s)$. What is the attenuation provided at 20 kHz?

Use the recipe on page 1094 of Sedra and Smith. Begin by determining ϵ,

$$\epsilon^2 = 10^{\frac{A_{\max}}{10}} - 1 = 10^{\frac{2}{10}} - 1 = 0.5849$$

Next determine the order of the filter by finding N which makes $A(\omega_s) \geq 15\,\text{dB}$ from the following formula

$$A(\omega_s) = 10 \log \left(1 + \epsilon^2 \left(\frac{\omega_s}{\omega_p} \right)^{2N} \right)$$

where $\omega_s/\omega_p = 1.5$. Here is a table

N	1	2	3	4	5
$A(\omega_s)$	3.6	6.0	8.8	12.0	15.4

so we need a 5th order filter. The radius of the poles is

$$f_0 = f_p \left(\frac{1}{\epsilon} \right)^{\frac{1}{N}} = 10 \times \left(\frac{1}{\sqrt{0.5849}} \right)^{\frac{1}{5}} = 10.551\,\text{kHz}$$

And thus $\omega_0 = 66.294 \times 10^3\,\text{s}^{-1}$. The angular spacing between the poles is $\Delta\phi = 180°/5 = 36°$. The phase of each pole is then $\phi_i = 108°$, $\phi_2 = 144°$, $\phi_3 = 180°$, $\phi_4 = 216°$, and $\phi_5 = 252°$, and the expressions for the poles is $p_i = \omega_0 (\cos\,\phi_i + j \sin\,\phi_i)$, so that

$$\begin{aligned}
p_1 &= -\,20.49 \times 10^3 + j63.05 \times 10^3\,\text{s}^{-1} \\
p_2 &= -\,53.63 \times 10^3 + j38.97 \times 10^3\,\text{s}^{-1} \\
p_3 &= -\,66.29 \times 10^3\,\text{s}^{-1} \\
p_4 &= -\,53.63 \times 10^3 + j38.97 \times 10^3\,\text{s}^{-1} \\
p_5 &= -\,20.49 \times 10^3 - j63.05 \times 10^3\,\text{s}^{-1}
\end{aligned}$$

The transfer function is then of course

$$T(s) = \frac{\omega_0^5}{(s-p_1)(s-p_2)(s-p_3)(s-p_4)(s-p_5)}$$

Where we can create p_3 from a first-ordre filter and pair p_1 and p_5 in a second order filter, and p_2 and p_4 in another second order filter.

The attenuation at 20 kHz is

$$A(20\,\text{kHz}) = 10 \log \left(1 + 0.5849 \times 2^{2\times 5}\right) = 27\,\text{dB}$$

SS 12.17. Contrast the attenuation provided by a fifth-order Chebyshev filter at $\omega_s = 2\omega_p$ to that provided by a Butterworth filter of equal order. For both, $A_{\max} = 1\,\text{dB}$. Sketch $|T|$ for both filters on the same axes.

For the Butterworth filter we have

$$|T(j\omega)| = \frac{1}{\sqrt{1+\epsilon^2\left(\frac{\omega}{\omega_p}\right)^{2N}}} \qquad \epsilon^2 = 10^{\frac{A_{\max}}{10}} - 1$$

Now for $A_{\max} = 1\,\mathrm{dB}$ we get $\epsilon^2 = 0.2590$, and for $\omega = 2\omega_p$ we get

$$|T(j2\omega)| = \frac{1}{\sqrt{1+0.2590\times 2^{10}}} = 0.061$$

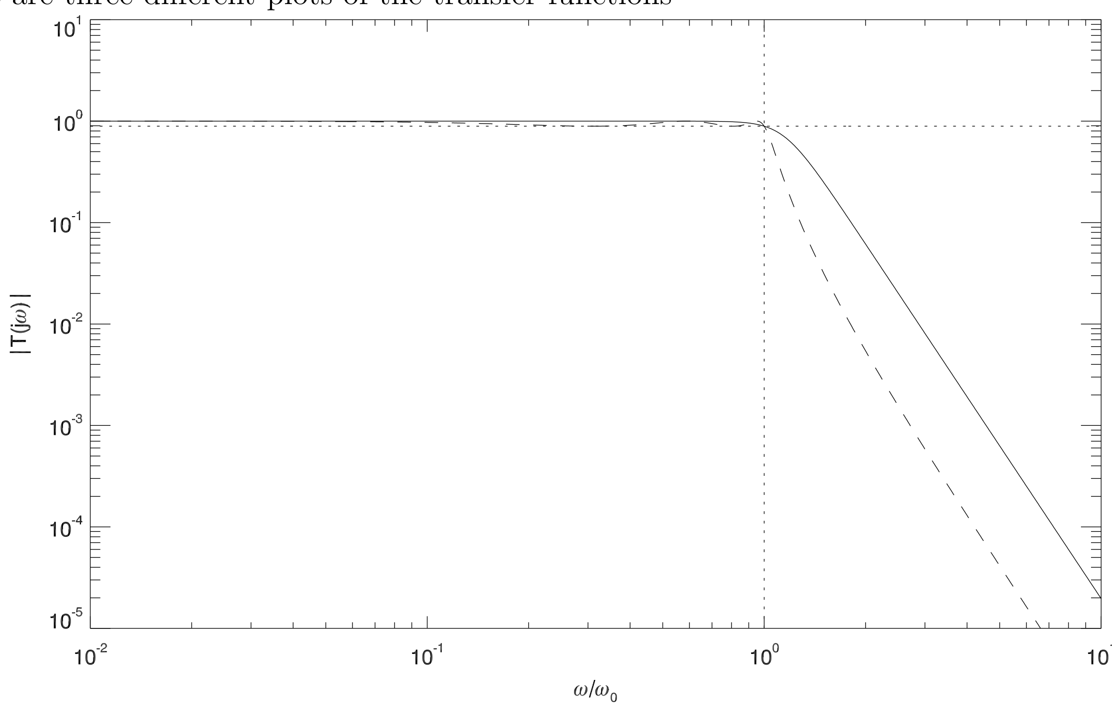
For the Chebyshev filter we have (for $\omega > \omega_p$)

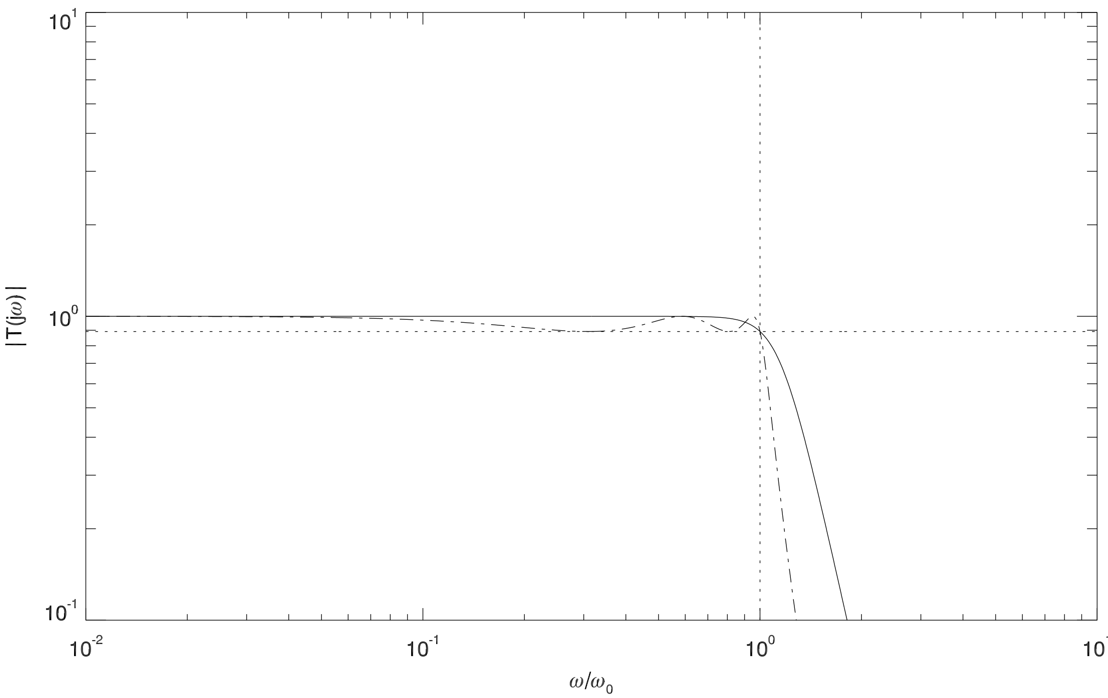
$$|T(j\omega)| = \frac{1}{\sqrt{1+\epsilon^2\cosh^2\left[N\cosh^{-1}\left(\frac{\omega}{\omega_p}\right)\right]}}$$

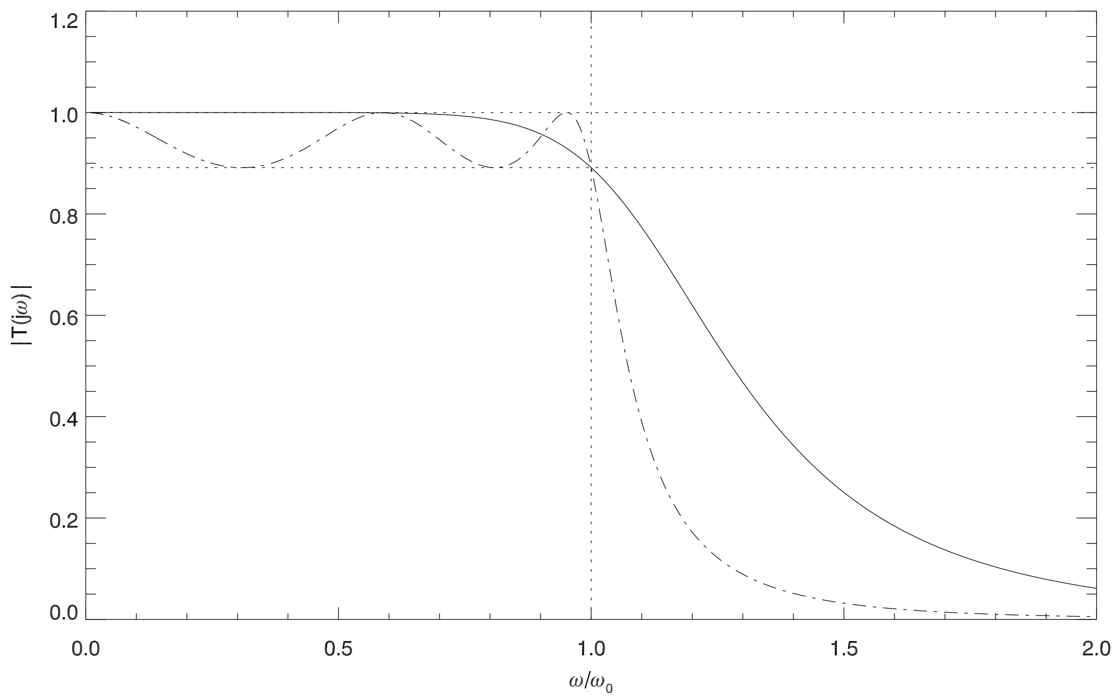
with the same expression for ϵ as before. Inserting we get

$$|T(j2\omega_p)| = \frac{1}{\sqrt{1+0.2590\cosh^2\left[5\cosh^{-1}2\right]}} = 0.00543$$

Here are three different plots of the transfer functions

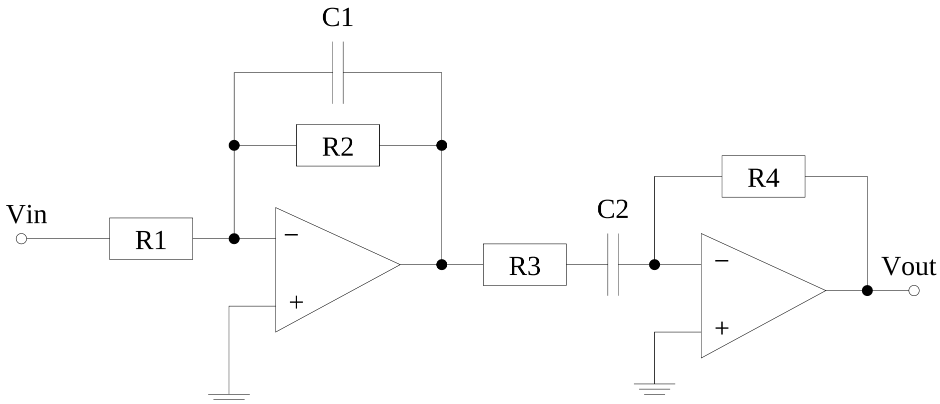






SS 12.22. By cascading a first-order op amp-RC low-pass circuit with a first-order op amp-RC high-pass circuit one can design a wideband bandpass filter. Provide such a design for the case in which the midband gain is 12 dB and the 3 − dB bandwidth extends from 100 Hz to 10 kHz. Select appropriate component values under the constraint that no resistors higher than 100 kΩ are to be used, and that the input resistance is to be as high as possible.

The circuit looks like this



The input resistance should be as large as possible. It is equal to R_1, $R_{\text{in}} = R_1$. To maximize the input resistance I will therefore select $R_1 = 100\,\text{k}\Omega$. I will also select $R_2 = 100\,\text{k}\Omega$ such that we get negative unity gain out of the low-pass filter. Next we want to select the resistors in the high-pass filter such that it produces a high-pass gain of 12 dB, which is a gain of a factor of 3.98. The gain of the high-pass filter is $-R_4/R_3$. I will select $R_4 = 100\,\text{k}\Omega$, and $R_3 = 25\,\text{k}\Omega$. Next we select the capacitors. We want the critical frequency of the low-pass filter to be $\omega_{\text{hi}} = 10\,\text{kHz}$, and the critical frequency of the high-pass filter to be $\omega_{\text{lo}} = 100\,\text{Hz}$. They are related to the component values as

$$R_2 C_1 = \frac{1}{\omega_{\text{hi}}} \qquad R_3 C_2 = \frac{1}{\omega_{\text{lo}}}$$

The capacitor values are then

$$C_1 = \frac{1}{\omega_{\text{hi}} R_2} = \frac{1}{2\pi 10 \times 10^3 \times 100 \times 10^3} = 1.6 \times 10^{-10} = 160\,\text{pF}$$

$$C_2 = \frac{1}{\omega_{\text{lo}} R_3} = \frac{1}{100 \times 25 \times 10^3} = 4.0 \times 10^{-7} = 400\,\text{nF}$$