

# EE 333 Electricity and Magnetism, Fall 2009

## Exam 1

**Rules:** This is a closed-book test. No calculator is needed for these problems, so a calculator is not allowed either. I expect you to know Maxwell's equations and basic math. If you need to be reminded of Maxwell's equations during the exam it will cost you 25% of your grade. Please let me know and I can provide you with a sheet. The exam will last 50 minutes, and each of the following numbered questions count equally toward your grade. None of these problems require extensive calculations. If you find yourself in long complex computations you are likely on the wrong track.

(1) What is the electric field corresponding to the following static volume charge distribution,  $\rho_v$ ?

$$\rho_v(r, \theta, \phi) = \begin{cases} \rho_0 & r \leq r_0 \\ \rho_0/2 & r_0 < r \leq r_1 \\ 0 & r_1 < r \end{cases}$$

The charge distribution is symmetric so the electric field must be radial, and we can write Gauss' law as

$$E_r = \frac{Q(r)}{4\pi\epsilon_0 r^2}$$

where  $Q(r)$  is the amount of charge contained inside a sphere of radius  $r$ .  
For  $r \leq r_0$  we get

$$Q = \frac{4\pi}{3} r^3 \rho_0$$

For  $r_0 < r \leq r_1$  we get

$$\begin{aligned} Q &= Q(r_0) + \left[ \frac{4\pi}{3} r^3 - \frac{4\pi}{3} r_0^3 \right] \frac{\rho}{2} \\ &= \frac{4\pi}{3} r_0^3 \rho_0 - \frac{4\pi}{3} r_0^3 \frac{\rho_0}{2} + \frac{4\pi}{3} r^3 \frac{\rho_0}{2} \\ &= \left[ \frac{4\pi}{6} r_0^3 + \frac{4\pi}{6} r^3 \right] \rho_0 \\ &= \frac{4\pi}{6} [r_0^3 + r^3] \rho_0 \end{aligned}$$

For  $r_1 < r$ , we get

$$Q(r) = Q(r_1) = \frac{4\pi}{6} [r_0^3 + r_1^3] \rho_0$$

The electric field is thus

$$E_r = \begin{cases} \frac{r\rho_0}{3\epsilon_0} & r \leq r_0 \\ \frac{r_0^3\rho_0}{6\epsilon_0 r^2} + \frac{r\rho_0}{6\epsilon_0} & r_0 < r \leq r_1 \\ \frac{r_0^3\rho_0}{6\epsilon_0 r^2} + \frac{r_1^3\rho_0}{6\epsilon_0 r^2} & r_1 < r \end{cases}$$

(2) Write an expression for what is causing the following electric field.

$$\vec{E}(\rho, \phi, z) = \hat{\phi} \frac{\rho}{\rho_0} E_0$$

$E_\phi$  does not depend on  $\phi$  and  $E_\rho = E_z = 0$  so the divergence of the field is zero. Therefore it cannot be the result of a charge distribution. Also, the line integral of the field around a circular path is non-zero.

$$2\pi\rho E_\phi = - \int_s \frac{\partial B_z}{\partial z} ds$$

$$2\pi \frac{\rho^2}{\rho_0} E_0 = - \int_s \frac{\partial B_z}{\partial z} ds$$

Note this is proportional to the area of the circle, so the integrand on the right must be constant,

$$2\pi E_0 \frac{\rho^2}{\rho_0} = -\pi \rho^2 \frac{\partial B_z}{\partial z}$$

$$\frac{\partial B_z}{\partial z} = -2 \frac{E_0}{\rho_0}$$

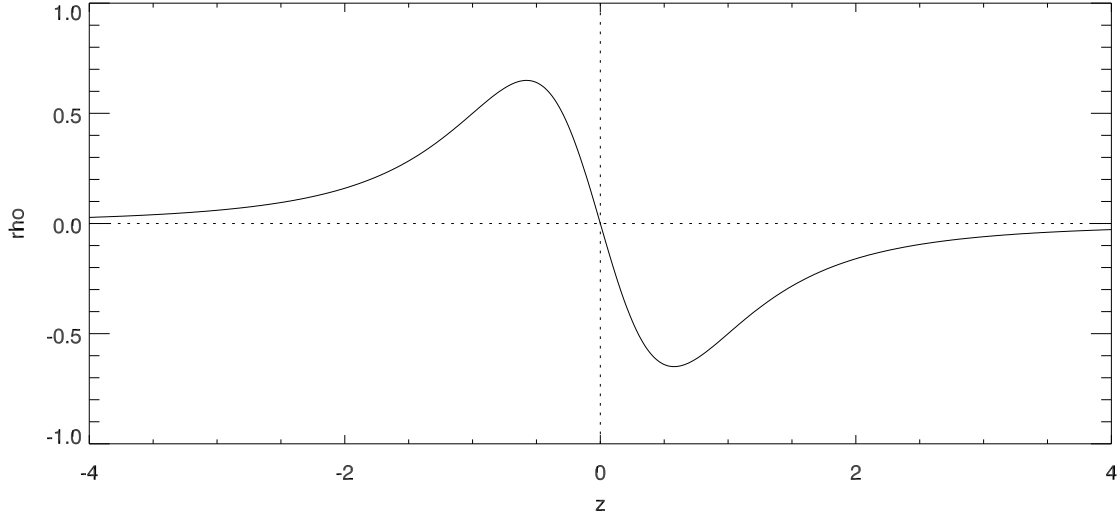
(3) Carefully sketch and label the charge distribution which is responsible for the following static electric field.

$$\vec{E}(x, y, z) = \hat{z} \frac{E_0}{1 + z^2}$$

Only the z-component is non-zero and it depends only on z, so we can write Gauss' law as

$$\begin{aligned} \rho &= \epsilon_0 \frac{\partial E_z}{\partial z} \\ &= \epsilon_0 \frac{\partial}{\partial z} \frac{E_0}{1 + z^2} \\ &= - \frac{2\epsilon_0 E_0 z}{(1 + z^2)^2} \end{aligned}$$

A plot (in arbitrary coordinates) is here:



(4) What static current distribution is responsible for the following magnetic field? Hint: use the integral form of Ampere's law, and explain why it must be a surface current (a current sheet), with units A/m.

$$\vec{B}(x, y, z) = \begin{cases} \hat{x}B_x & y > 0 \\ -\hat{x}B_x & y < 0 \end{cases}$$

The magnetic field changes discontinuously at the  $y = 0$  plane. This can only be caused by a current sheet in the  $y = 0$  plane for the following reason. Consider the static form of Ampere's law,

$$\oint_L \vec{B} \cdot d\vec{l} = \mu_0 \int_S \vec{J} \cdot d\vec{s}$$

Let the integration path consist of a segment in the  $y > 0$  region which is of length  $l$  and points in the same direction as the  $x$ -axis, and another segment in the  $y < 0$  region which is its mirror image and points opposite the  $x$ -axis. Each of those segments contribute  $lB_x$  to the path integral. The two connecting segments are perpendicular to the  $x$ -axis and contribute nothing. The surface vector for the flat surface bounded by this path points along the negative  $z$ -axis. The value of the path integral is independent of the  $y$ -coordinate of the two contributing line segments. That means all the current must be concentrated in the  $y = 0$  plane. The total amount of current is then  $J_s l$ , where  $J_s$  is the surface current magnitude in A/m, and  $l$  is the same  $x$ -extent as above. Thus we have

$$2lB_x = \mu_0 l J_s$$

or

$$J_s = \frac{2B_x}{\mu_0}$$

and it is directed in the negative  $z$  direction so we finally get

$$\vec{J}_s = -\hat{z} \frac{2B_x}{\mu_0} \text{ in the } y = 0 \text{ plane}$$