EE 333 Electricity and Magnetism, Fall 2009
Exam 3

Rules: This is a closed-book exam. You may use calculator, and the attached help sheets. The exam will last 50 minutes, and each of the following numbered questions count equally toward your grade. None of these problems require extensive calculations. If you find yourself in long complex computations your are likely on the wrong track.

Line charge

Consider a line charge along the z-axis with line charge density $\boldsymbol{\lambda}$, stretching from $z = -\frac{a}{2}$ to $z = \frac{a}{2}$.

1. **Show that the potential at a point along the positive z-axis, $z > \frac{a}{2}$, is**

$$\boldsymbol{\Phi(z) = \frac{\lambda}{4\pi\epsilon_0} \ln \frac{2z+a}{2z-a}}$$

The potential is

$$\begin{aligned}
\Phi =& \frac{1}{4\pi\epsilon_0} \int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{\lambda}{z-z'}\, dz' \\
=& \frac{\lambda}{4\pi\epsilon_0} \left[-\ln z - z'\right]_{-\frac{a}{2}}^{\frac{a}{2}} \\
=& \frac{\lambda}{4\pi\epsilon_0} \ln \frac{z+\frac{a}{2}}{z-\frac{a}{2}} \\
=& \frac{\lambda}{4\pi\epsilon_0} \ln \frac{2z+a}{2z-a}
\end{aligned}$$

2. **If $\boldsymbol{\lambda = 1\,\mu\mathrm{C/m}}$, $\boldsymbol{a = 1\,\mathrm{m}}$, compute the potential at $\boldsymbol{z = 0.55, 0.75, 1, 2, 5, 10, 20, 50\,\mathrm{m}}$.**

z (m)	0.55	0.75	1	2	5	10	20	50
Φ (V)	27.4×10^3	14.5×10^3	9.87×10^3	4.59×10^3	1.80×10^3	900	449	180

3. **Carefully show, by comparing new calculations to the above results, that the potential is as expected at large distance, $\boldsymbol{z \gg a}$.**

At large distance we expect the field to behave like the field of a point charge with charge $Q = \lambda a$, so we compare to the field

$$\Phi_0 = \frac{\lambda a}{4\pi\epsilon_0 z}$$

z (m)	0.55	0.75	1	2	5	10	20	50
Φ (V)	27.4×10^3	14.5×10^3	9.87×10^3	4.59×10^3	1.80×10^3	900	449	180
Φ_0 (V)	16.3×10^3	12.0×10^3	8.99×10^3	4.49×10^3	1.80×10^3	899	449	180

Notice that at large distance the two are equal.

Electrostatic energy

Consider a spherical shell of charge with uniform surface charge density $\boldsymbol{\sigma}$, and radius $\boldsymbol{a}$.

4. **Compute the electric potential, everywhere, due to this charge distribution (using the usual convention for zero potential).**

   Notice that for $r > a$ the field (and thus the potential) is the same as that of a point charge with the same total charge. Inside shell the electric field is zero and thus the potential is equal to the potential at radius a. So we have outside the sphere

   $$\Phi(r) = \frac{4\pi a^2 \sigma}{4\pi \epsilon_0 r} = \frac{a^2 \sigma}{\epsilon_0 r}$$

   inside the sphere we have

   $$\Phi(r) = \Phi(a) = \frac{a\sigma}{\epsilon_0}$$

   In summary we thus have

   $$\Phi(r) = \begin{cases} \frac{a\sigma}{\epsilon_0} & r \leq a \\ \frac{a^2\sigma}{\epsilon_0 r} & a < r \end{cases}$$

5. **How much energy did it take to assemble this charge distribution from infinity?**

   Every charge is at the potential $\Phi(a)$, so the total energy stored is simply

   $$W = \frac{Q\Phi(a)}{2} = 4\pi a^2 \sigma \frac{a\sigma}{\epsilon_0} \frac{1}{2} = \frac{2\pi a^3 \sigma^2}{\epsilon_0}$$