

EE 333 Electricity and Magnetism, Fall 2009 Homework #2 solution

1.26. Consider the contour c shown in Figure P1.26 and the vector field

$$\vec{F} = 2\rho(z^2 + 1)\cos\phi\hat{\rho} - \rho(z^2 + 1)\sin\phi\hat{\phi} + 2\rho^2z\cos\phi\hat{z}$$

(a) Evaluate $\int_c \vec{F} \cdot d\vec{l}$.

(b) Evaluate $\int_{c_1} \vec{F} \cdot d\vec{l}$, where c_1 is a straight line joining $(\rho = \rho_0, \phi = 0, z = 0)$ to $(\rho = 5, \phi = 0, z = 0)$.

(c) Are the results of (a) and (b) consistent with the field \vec{F} being conservative? (A field is conservative when its line integral along any closed contour is zero.)

(a) The line integral has five terms.

In the first term $d\vec{l} = \hat{\phi}d\phi$ and the integration range goes from $\phi = 0$ to $\phi = \frac{\pi}{3}$.

$$\begin{aligned} \text{term}_1 &= \int_0^{\frac{\pi}{3}} -25 \sin\phi d\phi \\ &= [25 \cos\phi]_0^{\frac{\pi}{3}} = 25(0.5 - 1) = -\frac{25}{2} \end{aligned}$$

It makes sense that it is negative because the $\hat{\phi}$ -component of the field points in the opposite direction to the path.

In the second term $d\vec{l} = \hat{z}dz$ and the integration range goes from $z = 0$ to $z = 5$.

$$\begin{aligned} \text{term}_2 &= \int_0^5 25z dz \\ &= \left[\frac{25}{2} z^2 \right]_0^5 \\ &= \frac{25^2}{2} = \frac{625}{2} \end{aligned}$$

It makes sense that it is positive because the \hat{z} -component of the vector field points in the direction of the path.

In the third term we have to be a little careful. The vector field $\hat{\rho}$ component points in the positive $\hat{\rho}$ direction. We can for example choose $d\vec{l} = -\hat{\rho}d\rho$ and the integration range from $\rho = \rho_0$ to $\rho = 5$.

$$\begin{aligned} \text{term}_3 &= \int_{\rho_0}^5 -26\rho d\rho \\ &= [-13\rho^2]_{\rho_0}^5 \\ &= -325 - (-13\rho_0^2) = 13\rho_0^2 - 325 \end{aligned}$$

Since $\rho_0 < 5$, this is negative as we expect it to be.

In the fourth term we once again have to be careful to get the sign right. The vector field points in the negative $\hat{\phi}$ direction, so we expect a positive sign for the integral. We could for example choose $d\vec{l} = -\hat{\phi} \rho d\phi$ and the integration range from $\phi = 0$ to $\phi = \frac{\pi}{3}$.

$$\begin{aligned} \text{term}_4 &= \int_0^{\frac{\pi}{3}} 26\rho_0^2 \sin \phi d\phi \\ &= [-26\rho_0^2 \cos \phi]_0^{\frac{\pi}{3}} \\ &= -26\rho_0^2 (0.5 - 1) \\ &= 13\rho_0^2 \end{aligned}$$

which is positive just as we expect.

In the fifth term $d\vec{l} = -\hat{z} dz$ and the integration range goes from $z = 0$ to $z = 5$ in order to get the sign right. Since the vector field has a component along the positive \hat{z} direction, we expect the result to be negative.

$$\begin{aligned} \text{term}_5 &= \int_0^5 -2\rho_0^2 z dz \\ &= [-\rho_0^2 z^2]_0^5 \\ &= -25\rho_0^2 \end{aligned}$$

And it is negative as we expect.

The line integral along the entire path is then

$$\begin{aligned} \text{total} &= -\frac{25}{2} + \frac{625}{2} + 13\rho_0^2 - 325 + 13\rho_0^2 - 25\rho_0^2 \\ &= -25 + \rho_0^2 \end{aligned}$$

(b) The path integral along the connecting path is then computed by setting $d\vec{l} = \hat{\rho} d\rho$, and the integration range from $\rho = \rho_0$ to $\rho = 5$.

$$\begin{aligned} \text{short} &= \int_{\rho_0}^5 2\rho d\rho \\ &= [\rho^2]_{\rho_0}^5 = 25 - \rho_0^2 \end{aligned}$$

(c) The sum of the path “total” and the path “short” is a closed loop integral, and equal to zero, so these line integrals are consistent with the vector field \vec{F} being conservative.

1.31. Determine the net flux of the vector field $\vec{F}(r, \theta, \phi) = r \sin \theta \hat{r} + \hat{\theta} + \hat{\phi}$ emanating from a closed surface defined by $r = 1$, $0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq \phi \leq 2\pi$. hint: The closed surface consists of the hemisphere s_1 and the base plane s_2 shown in Figure P1.31.

The surface element on the sphere is $r^2 \sin \theta d\theta d\phi \hat{\rho}$, so only the $\hat{\rho}$ -component will contribute to the surface integral over the sphere. We write, setting $r = 1$,

$$\begin{aligned}
\text{sphere term} &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{2}} \sin^2 \theta \, d\theta \, d\phi \\
&= 2\pi \int_0^{\frac{\pi}{2}} \sin^2 \theta \, d\theta \\
&= 2\pi \left[\frac{\theta}{2} - \frac{1}{4} \sin 2\theta \right]_0^{\frac{\pi}{2}} \\
&= 2\pi \frac{\pi}{4} = \frac{\pi^2}{2}
\end{aligned}$$

For the case of the plane term the surface element is $-\hat{z}dxdy$ in cartesian coordinates, and in spherical coordinates it is $\hat{\theta}\rho d\phi d\rho$. Thus, only the $\hat{\theta}$ component contributes, and it is constant over the $\theta = \frac{\pi}{2}$ surface. Thus we can compute the integral simply as the area times the magnitude of the vector, which is 1, so

$$\text{plane term} = \pi r^2 = \pi$$

The total flux of the field \vec{F} out of the surface given is thus

$$\int_S \vec{F} \cdot d\vec{s} = \frac{\pi^2}{2} + \pi$$

1.34. Consider two concentric cylindrical surfaces shown in Figure P1.34, one having a radius a and a charge density ρ_s , and the other having a radius b and a charge density $-\rho_s$. Find the electric field \vec{E} in the following regions:

- (a) $\rho < a$.
- (b) $a < \rho < b$.
- (c) $\rho > b$.

I am going to use Gauss' law for electric fields for this problem,

$$\int_S \vec{E} \cdot d\vec{s} = \frac{1}{\epsilon_0} \int_V \rho \, dV$$

and integrating over cylindrical surfaces. Due to symmetry arguments the electric field must point in the $\hat{\rho}$ direction and have the same magnitude all over the surface. Thus we can compare the integral of the electric field per unit length of the cylinder to the charge per unit length of the cylinder,

$$2\pi\rho E_\rho = \frac{1}{\epsilon_0} \sum_{i=0}^N 2\pi\rho_i \rho_{si}$$

or

$$E_\rho = \frac{1}{\epsilon_0\rho} \sum_{i=0}^N \rho_i \rho_{si}$$

where the sum is over the N surfaces which have $\rho_i < \rho$.

(a) In this case there are no surfaces inside ρ , so $E_\rho = 0$ for $\rho < a$.

(b) In this case there is one surface at a , so we get

$$E_\rho = \frac{\rho_s a}{\epsilon_0 \rho}$$

(c) In this case there are two surfaces, the first at a with $\rho_{si} = \rho_s$, and the second at b with $\rho_{si} = -\rho_s$, so we get

$$\begin{aligned} E_\rho &= \frac{\rho_s a}{\epsilon_0 \rho} - \frac{\rho_s b}{\epsilon_0 \rho} \\ &= \frac{\rho_s}{\epsilon_0 \rho} (a - b) \end{aligned}$$