EE 333 Electricity and Magnetism, Fall 2009
Homework #3 solution

2.21. Which of the following vectors can be a static electric field? Determine the charge density associated with it.
(a)

$$\vec{\boldsymbol{E}} = \boldsymbol{a x^2 y^2 \hat{x}}, \quad \boldsymbol{a} \text{ is constant}$$

(b)

$$\vec{\boldsymbol{E}} = \frac{\boldsymbol{a}}{\boldsymbol{\rho^2}} \left[\hat{\boldsymbol{\rho}} \left(\mathbf{1} + \cos \boldsymbol{\phi}\right) + \hat{\boldsymbol{\phi}} \sin \boldsymbol{\phi} \right]$$

If we look at Gauss' law for $\vec{E}$ and Faraday's law, we find that the divergence of the charge density is associated with charge distribution whereas its curl is associated with a changing magnetic field. A static electric field cannot have a non-zero curl. So let's begin by computing the curl of the two fields, and then compute the charge density for the one which has zero curl. (It is also perfectly valid to get the charge density for the one which has non-zero curl, and that charge density will be associated with the static component of the electric field. The problem just doesn't ask us for that.)
(a) In cartesian coordinates

$$\nabla \times \vec{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) \hat{x} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) \hat{y} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \hat{z}$$

Since $\vec{E} = \hat{x} E_x$, the curl can have components in the $\hat{y}$ and $\hat{z}$ directions, but not in the $\hat{x}$ direction. We see that

$$\frac{\partial E_x}{\partial z} = 0$$

and

$$\frac{\partial E_x}{\partial y} = 2ax^2 y$$

Thus

$$\nabla \times \vec{E} = -2ax^2 y \hat{z}$$

which is non-zero. This electric field therefore cannot represent a static situation.
(b) In cyllindrical coordinates

$$\nabla \times \vec{E} = \left(\frac{1}{\rho} \frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z} \right) \hat{\rho} + \left(\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} \right) \hat{\phi} + \frac{1}{\rho} \left(\frac{\partial \left(\rho E_\phi\right)}{\partial \rho} - \frac{\partial E_\rho}{\partial \phi} \right) \hat{z}$$

The $\hat{\rho}$ and the $\hat{\phi}$ components are non-zero, and each depend only on ρ and ϕ. In other words, only two derivatives can be non-zero

$$\frac{\partial E_\rho}{\partial \phi} = -\frac{a}{\rho^2} \sin \phi$$

$$\frac{\partial\left(\rho E_\phi\right)}{\partial\rho}=\frac{\partial}{\partial\rho}\left(\frac{a}{\rho}\sin\phi\right)=-\frac{a}{\rho^2}\sin\phi$$

Inserting we find that $\nabla\times\vec{E}=0$. Thus, this field is completely determined by a static charge distribution. Next we compute the charge density associated with this,

$$\nabla\cdot\vec{E}=\frac{1}{\rho}\frac{\partial\left(\rho E_\rho\right)}{\partial\rho}\hat{\rho}+\frac{1}{\rho}\frac{\partial E_\phi}{\partial\phi}\hat{\phi}+\frac{\partial E_z}{\partial z}\hat{z}$$

only the first two can be non-zero, so

$$\frac{\partial\left(\rho E_\rho\right)}{\partial\rho}=\frac{\partial}{\partial\rho}\left[\frac{a}{\rho}\left(1+\cos\phi\right)\right]=-\frac{a}{\rho^2}\left(1+\cos\phi\right)$$

and

$$\frac{\partial E_\phi}{\partial\phi}=\frac{\partial}{\partial\phi}\left[\frac{a}{\rho^2}\sin\phi\right]=\frac{a}{\rho^2}\cos\phi$$

Putting it together (inserting) we get

$$\begin{aligned}\rho=&\epsilon_0\nabla\cdot\vec{E}=-\frac{\epsilon_0 a}{\rho^3}\left(1+\cos\phi\right)+\frac{\epsilon_0 a}{\rho^3}\cos\phi\\=&-\frac{\epsilon_0 a}{\rho^3}\end{aligned}$$

This can't be right. There must be a ϕ-dependence on the charge density.

2.23. Given a cylindrical electron beam of radius a, and if the electric field inside the beam is given by $\vec{E}=\left(\rho_0/4\pi\epsilon_0 a^2\right)\rho^3\hat{\rho}$ where ρ is a constant, find the charge density in the beam. Also if the magnetic field inside the beam is given by $\vec{B}=\left(\mu_0 J_0/3a\right)\rho^2\hat{\phi}$ where J_0 is a constant, determine the current density $\vec{J}$.

First the charge density. I will use the differential form of Gauss' law for electric fields,

$$\begin{aligned}\rho=&\epsilon_0\nabla\cdot\vec{E}=\epsilon_0\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\rho_0}{4\pi\epsilon_0 a^2}\rho^3\right]\\=&\epsilon_0\frac{1}{\rho}\frac{4\rho_0}{4\pi\epsilon_0 a^2}\rho^3\\=&\frac{\rho_0}{\pi a^2}\rho^2\end{aligned}$$

This is only valid for $\rho<a$.

Next for the current density

$$\vec{J}=\frac{1}{\mu_0}\nabla\times\vec{B}-\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

but since the given $\vec{E}$ does not depend on time we get

$$\vec{J} = \frac{1}{\mu_0} \nabla \times \vec{B}$$

Only one term is non-zero:

$$\begin{aligned}
\vec{J} =& \frac{1}{\mu_0 \rho} \frac{\partial \rho B_\phi}{\partial \rho} \\
=& \frac{1}{\mu_0 \rho} \frac{\partial}{\partial \rho} \left[\rho \frac{\mu_0 J_0}{3a} \rho^2 \right] \\
=& \frac{1}{\mu_0 \rho} \frac{3\mu_0 J_0}{3a} \rho^2 \\
=& J_0 \frac{\rho}{a}
\end{aligned}$$

Again, this is only valid for $\rho < a$.

2.34. The electric field intensity of a uniform plane wave is given by

$$\vec{\boldsymbol{E}} = 15 \cos\left(\pi \times 10^6 t + \frac{\pi}{3} z\right) \hat{y} \quad \mathrm{V/m}$$

(a) Direction (polarization) of electric field.
(b) Direction of propagation.
(c) Frequency and wavelength.
(d) Magnetic field intensity vector $\vec{H}$. Specifically indicate the direction of $\vec{H}$.

(a) This wave is polarized in the y-direction.
(b) The direction of propagation is the negative z-axis (because the coefficients of the t and z term are of the same sign).
(c) We see that $\omega = 10^6\,\mathrm{s}^{-1}$, and $\beta = \frac{\pi}{3}$, so that

$$f = \frac{\omega}{2\pi} = \frac{\pi 10^6}{2\pi} = 500 \times 10^3$$

$$\lambda = \frac{2\pi}{\beta} = 6$$

The units for ω and β are not given in the original expression so I cannot specify units on my answers either.
(d) We know that

$$\frac{E}{H} = \eta_0$$

such that

$$H = \frac{E}{\eta_0} = \frac{15}{377} = 0.040\,\mathrm{A/m}$$

The direction should be such that the cross product $\vec{E} \times \vec{H}$ points in the direction of propagation. So if the E-field points in the y-direction and the direction of propagation in the negative z-direction, the magnetic intensity field must point in the positive x-direction. Thus, the expression for the magnetic field is

$$\vec{E} = 0.040 \cos\left(\pi \times 10^6 t + \frac{\pi}{3} z\right) \hat{x} \ \ \text{A/m}$$