EE 333 Electricity and Magnetism, Fall 2009
Homework #4 solution

1.37. In Figure P1.37 a spherical cloud of charge in free space is characterized by a volume charge density

$$\boldsymbol{\rho_v} = \begin{cases} \boldsymbol{\rho_0}\left(\mathbf{1} - \frac{\boldsymbol{r^2}}{\boldsymbol{a^2}}\right) & \boldsymbol{r < a} \\ \mathbf{0} & \boldsymbol{r \geq a} \end{cases}$$

The cloud is symmetric, so the we have right away that

$$E_r = \frac{Q(r)}{4\pi\epsilon_0 r^2}$$

and

$$Q(r) = \int_0^r \rho_v 4\pi r^2 dr$$

For $r \leq a$:

$$\begin{aligned} Q(r) =& 4\pi\rho_0 \int_0^r \left(r^2 - \frac{r^4}{a^2}\right) dr \\ =& 4\pi\rho_0 \left[\frac{r^3}{3} - \frac{r^5}{5a^2}\right]_0^r \\ =& 4\pi\rho_0 \left(\frac{r^3}{3} - \frac{r^5}{5a^2}\right) \end{aligned}$$

For $r > a$ we have

$$\begin{aligned} Q(r) =& Q(a) \\ =& 4\pi\rho_0 \left(\frac{a^3}{3} - \frac{a^5}{5a^2}\right) \\ =& 4\pi\rho_0 a^3 \left(\frac{1}{3} - \frac{1}{5}\right) \\ =& 4\pi\rho_0 a^3 \frac{2}{15} \end{aligned}$$

The final expression for E_r is then

$$E_r = \begin{cases} \frac{\rho_0}{\epsilon_0}\left(\frac{r}{3} - \frac{r^3}{a^2}\right) & r \leq a \\ \frac{\rho_0}{\epsilon_0}\frac{2}{15}\frac{a^3}{r^2} \end{cases}$$

1.47. (a) In a region of space in the neighborhood of an electromagnetic plastic heat sealer, the magnetic field of the source is unknown and is assumed to be

arbitrarily oriented. Suggest a practical procedures to measure this arbitrarily oriented magnetic field.
(b) The magnetic field intensity $\vec{B}$ of a short electric current source may be approximately given in the cylindrical coordinates by

$$\vec{B} = \left(K_1 \frac{1}{\rho^2} - \frac{K^2}{\rho} \right) \sin \omega t \hat{\phi}$$

To measure this magnetic field, the rectangular conducting loop shown in Figure P1.47 is placed in the y-z plane.
(i) Calculated the induced EMF at the terminals of the conducting loop.
(ii) Show that the variation of the induced EMF and the magnetic flux satisfy Lenz's law.
(a) A good procedure for measuring a magnetic field is with a small wire loop over which we measure the electric voltage, induced EMF. In that case we have by Faraday's law that the voltage along one turn of the loop is equal to the area of the loop times the time-derivative of the magnetic field perpendicular to the loop,

$$V = A \frac{\partial B_\perp}{\partial t}$$

If the magnetic field is harmonic with frequency ω, for example

$$\vec{B} = \vec{B}_0 \sin \omega t$$

we get

$$V_0 = A \omega B_{0\perp}$$

where V_0 is the amplitude of the EMF and $\vec{B}_{0\perp}$, is the amplitude of the magnetic field perpendicular to the loop. Orient the loop in three different mutually perpendicular directions to measure the vector magnetic field.
(b)
(i) I use Faraday's law,

$$\oint_L \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot d\vec{s}$$

In this case, the line integral of the electric field is the measured EMF, and the magnetic field is oriented in the ϕ-direction such that we can carry out a scalar integral of the magnitude of $\vec{B}$. I also choose the direction of integration around the loop to be such that the surface vector is oriented in the $\hat{\phi}$ direction. This makes the terminal further from the z-axis the positive terminal for a postive $\frac{\partial \vec{B}}{\partial t}$.

$$
\begin{aligned}
V =& -\frac{\partial}{\partial t}\int_S B_\phi ds \\
=& -\omega\cos\omega t\ a\int_{\rho_1}^{\rho_1+b}\left(\frac{K_1}{\rho^2}-\frac{K^2}{\rho}\right)d\rho \\
=& -\omega\cos\omega t\left[-\frac{K_1}{\rho}-K^2\ln\rho\right]_{\rho_1}^{\rho_1+b} \\
=& \omega\cos\omega t\left[K_1\left(\frac{1}{\rho_1+b}-\frac{1}{\rho_1}\right)+K^2\ln\frac{\rho_1+b}{\rho_1}\right]
\end{aligned}
$$

(ii) If the $\frac{\partial\vec{B}}{\partial t}$ points in the $\hat{\phi}$ direction, the electric field will point in the negative right-hand direction around the $\hat{\phi}$ direction (because of the '-' sign in Faraday's alw). That EMF will attempt to drive a current through the wire. According to Faraday's law that current will create a magnetic field pointed in the right-hand direction around the direction of the current. That direction is in the $-\hat{\phi}$ direction. Therefore the fields and EMF satisfy Lenz's law.

2.26. (a) The vector $\vec{E}$ expressed in the cylindrical coordinate system

$$\vec{E} = 3\rho^2\hat{\rho} + \rho\cos\phi\hat{\phi} + \rho^3\hat{z}$$

represents a static electric field. Calculate the volume charge density associated with this electric field at the point $(0.5, \frac{\pi}{3}, 0)$. (b) If the vector $\vec{B}$ represents a magnetic flux density in free space,

$$\vec{B} = B_r\hat{r} + \sin\theta\cos\phi\hat{\theta} + r\sin\phi\hat{\phi}$$

find the component B_r of this vector. (Specify an integration constant such that B_r remains finite as $r \to 0$).

(a) We use Gauss' law for electric fields,

$$\rho = \epsilon_0\nabla\cdot\vec{E}$$

which in cylindrical coordinates is

$$
\begin{aligned}
\rho =& \epsilon_0\left[\frac{1}{\rho}\frac{\partial\rho E_\rho}{\partial\rho}+\frac{1}{\rho}\frac{\partial E_\phi}{\partial\phi}+\frac{\partial E_z}{\partial z}\right] \\
=& \epsilon_0\left[9\rho-\sin\phi\right]
\end{aligned}
$$

(b) We use Gauss' law for magnetic fields,

$$\nabla\cdot\vec{B} = 0$$

which then fixes the relationship between the three components of the magnetic field. In spherical coordinates we get

$$
\begin{aligned}
\nabla\cdot\vec{B} =& \frac{1}{r^2}\frac{\partial r^2 B_r}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(B_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial B_\phi}{\partial\phi}\\
=& \frac{1}{r^2}\frac{\partial r^2 B_r}{\partial r} + \frac{1}{r\sin\theta}2\sin\theta\cos\theta\cos\phi + \frac{1}{r\sin\theta}r\cos\phi\\
=& \frac{1}{r^2}\frac{\partial r^2 B_r}{\partial r} + \frac{2}{r}\cos\theta\cos\phi + \frac{\cos\phi}{\sin\theta}
\end{aligned}
$$

Because $\nabla\cdot\vec{B} = 0$ we then write

$$
\frac{1}{r^2}\frac{\partial r^2 B_r}{\partial r} = -\frac{2}{r}\cos\theta\cos\phi - \frac{\cos\phi}{\sin\theta}
$$

or

$$
\frac{\partial r^2 B_r}{\partial r} = -2r\cos\theta\cos\phi - r^2\frac{\cos\phi}{\sin\theta}
$$

Integrating we get

$$
r^2 B_r = -r^2\cos\theta\cos\phi - \frac{r^3}{3}\frac{\cos\phi}{\sin\theta} + C
$$

Or

$$
B_r = -\cos\theta\cos\phi - \frac{r\cos\phi}{3\sin\theta} + \frac{C}{r^2}
$$

I will specify C such that $C/r^2 \to 0$ as $r \to 0$ ($C = 0$). Thus

$$
\begin{aligned}
B_r =& -\cos\phi\cos\theta - \frac{r\cos\phi}{3\sin\theta}\\
=& -\cos\phi\left(\cos\theta - \frac{r}{3\sin\theta}\right)
\end{aligned}
$$

2.35. The superposition of two uniform plane waves of equal magnitudes and propagating in opposite directions results in a composite wave having electric and magnetic fields given by

$$
\begin{aligned}
\vec{\boldsymbol{E}}(z,t) =& 2E_m \sin\beta_0 z \sin\omega t\,\hat{\boldsymbol{x}}\\
\vec{\boldsymbol{H}}(z,t) =& 2\frac{E_m}{\eta_0}\cos\beta_0 z\cos\omega t\,\hat{\boldsymbol{y}}
\end{aligned}
$$

Show that these fields satisfy Maxwell's equations and the scalar wave equations for time-harmonic electric and magnetic fields.

First we note that it is consistent with the two version of Gauss' law. The non-zero components do not depend on their corresponding coordinate, so the divergence must be zero. Next, check Faraday's law.

$$\nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$$

The only non-zero derivative in the curl is $\frac{\partial E_x}{\partial z}$, which appears in the $\hat{y}$ component of the curl.

$$-\frac{\partial E_x}{\partial z}\hat{y} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$$

$$-2E_m\beta_0 \cos\beta_0 z\, \sin\omega t\, \hat{y} = -2\mu_0 \frac{E_m}{\eta_0}\omega \cos\beta_0 z\, \sin\omega t\, \hat{y}$$

$$\beta_0 = \mu_0 \frac{\omega}{\eta_0}$$

$$\frac{\omega}{\beta_0} = \frac{\eta_0}{\mu_0} = \sqrt{\frac{\mu_0}{\epsilon_0}}\frac{1}{\mu_0} = \frac{1}{\sqrt{\mu_0\epsilon_0}}$$

and that is what we expect. Now for Ampere's law,

$$\nabla \times \vec{B} = \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

or, since $\vec{B} = \mu_0\vec{H}$,

$$\nabla \times \vec{H} = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Only the $\frac{\partial H_y}{\partial z}$ derivative is non-zero. It is part of the $\hat{x}$ component of the curl,

$$-\frac{\partial H_y}{\partial z} = \epsilon_0 \frac{\partial E_x}{\partial t}$$

$$2\beta_0 \frac{E_m}{\eta_0} \sin\beta_0 z \cos\omega t = 2\epsilon_0\omega E_m \sin\beta_0 z \cos\omega t$$

$$\frac{\beta_0}{\eta_0} = \epsilon_0\omega$$

$$\frac{\omega}{\beta_0} = \frac{1}{\eta_0\epsilon_0} = \frac{1}{\sqrt{\frac{\mu_0}{\epsilon_0}}\epsilon_0} = \frac{1}{\sqrt{\mu_0\epsilon_0}}$$

and that is what we expect.