EE 333 Electricity and Magnetism, Fall 2009
Homework #5 solution

2.17. Show that the vector

$$\vec{B} = \frac{1}{r^2} \sin \phi \cos^2 \theta \hat{r}$$

may represent a static magnetic flux density vector. Determine the current density associated with it.

The magnetic field is a valid static magnetic field if has zero divergence. Note only the radial component of the field is non-zero

$$
\begin{aligned}
\nabla \cdot \vec{B} &= \left(\nabla \cdot \vec{B} \right)_r \\
&= \frac{1}{r^2} \frac{\partial r^2 B_r}{\partial r} \\
&= \frac{1}{r^2} \frac{\partial}{\partial r} \sin \phi \cos^2 \theta \\
&= 0
\end{aligned}
$$

The field thus has zero divergence and obeys Gauss' law for magnetic fields. We find the current distribution from Ampere's law. Because the magnetic field only has a radial component, only the terms $\frac{\partial B_r}{\partial \phi}$ and $\frac{\partial B_r}{\partial \theta}$ mayh be non-zero. Thus,

$$
\begin{aligned}
\vec{J} &= \frac{1}{\mu_0} \nabla \times \vec{B} \\
&= \frac{1}{\mu_0} \left[\frac{1}{r \sin \theta} \frac{\partial B_r}{\partial \phi} \hat{\theta} - \frac{1}{r} \frac{\partial B_r}{\partial \theta} \hat{\phi} \right] \\
&= \frac{1}{\mu_0} \left[\frac{1}{r \sin \theta} \frac{1}{r^2} \cos \phi \cos^2 \theta \, \hat{\theta} + \frac{1}{r} \frac{1}{r^2} \sin \phi \, 2 \cos \theta \, \sin \theta \hat{\phi} \right] \\
&= \frac{1}{\mu_0 r^3} \left[\frac{\cos \phi \cos^2 \theta}{\sin \theta} + 2 \sin \phi \cos \theta \sin \theta \right]
\end{aligned}
$$

2.19. The vector $\vec{E}$ expressed in the cyllindrical coordinate system by

$$\vec{E} = 3\rho^2 \, \hat{\rho} + \rho \cos \phi \, \hat{\phi} + \rho^3 \, \hat{z}$$

represents a static electric field. Calculate the volume charge density associated with this electric field at the point $(0.5, \pi/3, 0)$.

We use Gauss' law for electric fields,

$$\rho = \epsilon_0 \nabla \cdot \vec{E}$$

which in cylindrical coordinates is

$$\rho = \epsilon_0 \left[\frac{1}{\rho} \frac{\partial \rho E_\rho}{\partial \rho} + \frac{1}{\rho} \frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z} \right]$$
$$= \epsilon_0 \left[9\rho - \sin\phi \right]$$

3.1. The electric field $\vec{E} = 3z^2 y \cos(10^8 t)\,\hat{x}$ is applied to a dielectric material (Lucit) of $\epsilon_r = 2.56$. Determine the following:

   (a) The electric polarization.

   (b) The induced polarization charge density ρ_p. Explain physically the reason for the zero value of ρ_p.

   (c) The polarization current density J_p.

(a) The electric polarization is related to the electric field as

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

where $\chi_e = \epsilon_r - 1 = 1.56$. Thus,

$$\vec{P} = 3 \times 1.56\epsilon_0 z^2 y \cos\left(10^8 t\right)\hat{x}$$
$$= 4.68 z^2 y \cos\left(10^8 t\right)\hat{x}$$

(b) The polarization charge density is related to the polarization through a divergence. In this case the polarization only points in the $\hat{x}$ direction so we only need to take the x-derivative.

$$\rho_p = -\nabla \cdot \vec{P}$$
$$= \frac{\partial P_x}{\partial x}$$
$$= 0$$

This answer makes sense. The polarization is constant along the direction of the polarization. Therefore, for every charge shifted out of a region, an equal charge is shifted into the region.
(c) The polarization current density is

$$\vec{J}_p = \frac{\partial \vec{P}}{\partial t}$$
$$= -3z^2 y \times 10^8 \times \sin\left(10^8 t\right)$$
$$= -3 \times 10^8 z^2 y \sin\left(10^8 t\right)$$

3.2. A coaxial power cable has a core (conductor) of radius a. The region between the inner and outer conductors is filled with two concentric layers of dielectrics, $\epsilon_1 = 1.5\epsilon_0$, and $\epsilon_2 = 4.5\epsilon_0$ as shown in Figure P3.2. If the outer conductor is grounded while the inner conductor is raised to a voltage that produces a linear charge density distribution ρ_l, determine the following:

2

(a) **The electric flux density, the electric field intensity, and the polarization in the two regions inside and the air outside the cable.**

(b) **The polarization surface charge at $\rho = a$, and $\rho = r_1$.**

(c) **The polarization charge density in region 2.**

The inner conductor is charged to a charge of ρ_l per unit length of the inner conductor. The electric field is radial due to the symmetry of the problem.

(a) The electric flux density is the $\vec{E}$ vector. We use Gauss' law for electric fields in integral form,

$$\oint_S \vec{D} \cdot d\vec{s} = \int_V \rho \, dV$$

Make the surface be a cylinder of fixed radis and length l. As long as the volume contains the inner conductor, but not the outer conductor, then

$$\int_V \rho \, dV = l\rho_l$$

If it contains the outer conductor the RHS is zero.
The surface integral is

$$\oint_S \vec{D} \cdot d\vec{s} = 2\pi r l D_\rho$$

Overall we then have (eliminate l)

$$D_\rho = \begin{cases} \frac{\rho_l}{2\pi\rho} & \text{in region 1 and 2} \\ 0 & \text{outside outer conductor} \end{cases}$$

We also have $D_\rho = \epsilon E_\rho$, so $E_\rho = \frac{D_\rho}{\epsilon_r \epsilon_0}$. And since $P_\rho = \chi_e \epsilon_0 E_\rho$, we can see that

$$P_\rho = \chi_e \epsilon_0 \frac{D_\rho}{\epsilon_r \epsilon_0} = \frac{\epsilon_r - 1}{\epsilon_r} D_\rho = \left(1 - \frac{1}{\epsilon_r}\right) D_\rho$$

Now we are ready to do the problem. In region 1,

$$D_\rho = \frac{\rho_l}{2\pi\rho}$$

$$E_\rho = \frac{D_\rho}{\epsilon_r \epsilon_0} = \frac{\rho_l}{2\pi\epsilon_r\epsilon_0\rho} = \frac{\rho_l}{3\pi\epsilon_0\rho}$$

$$P_\rho = \left(1 - \frac{1}{\epsilon_r}\right) D_\rho = \left(1 - \frac{1}{1.5}\right) \frac{\rho_l}{2\pi\rho} = \frac{1}{3}\frac{\rho_l}{2\pi\rho} = \frac{\rho_l}{6\pi\rho}$$

Note that

$$\epsilon_0 E_\rho + P_\rho = \frac{\rho_l}{3\pi r} + \frac{\rho_l}{6\pi r} = \frac{\rho_l}{\pi r}\left(\frac{1}{3} + \frac{1}{6}\right) = \frac{\rho_l}{\pi r}\frac{1}{2} = \frac{\rho_l}{2\pi r} = D_\rho$$

as we expect.

In region 2,

$$D_\rho = \frac{\rho_l}{2\pi\rho}$$

$$E_\rho = \frac{D_\rho}{\epsilon_r \epsilon_0} = \frac{\rho_l}{2\pi\epsilon_r\epsilon_0\rho} = \frac{\rho_l}{2 \times 4.5\epsilon_0\rho} = \frac{\rho_l}{9\pi\epsilon_0\rho}$$

$$P_\rho = \left(1 - \frac{1}{\epsilon_r}\right) D_\rho = \left(1 - \frac{1}{4.5}\right) \frac{\rho_l}{2\pi\rho} = \frac{7}{9}\frac{\rho_l}{2\pi\rho} = \frac{7\rho_l}{18\pi\rho}$$

Note that

$$\epsilon_0 E_\rho + P_\rho = \frac{\rho_l}{\pi\rho}\left(\frac{1}{9} + \frac{7}{18}\right) = \frac{\rho_l}{\pi\rho}\left(\frac{2}{18} + \frac{7}{18}\right) = \frac{\rho_l}{\pi\rho}\frac{1}{2} = \frac{\rho_l}{2\pi\rho} = D_\rho$$

as we expect.

Outside the cable we have $D_\rho = 0$, so $E_\rho = 0$ and $P_\rho = 0$ because of the proportionality between the three quantities.

(b) At the interface between the inner conductor and region 1 the polarzation changes discontinuously from $P_\rho = 0$ in the conductor (because there is no electric field in the conductor and the polarization is proporational to the electric field), to $P_\rho = \frac{\rho_l}{6\pi a}$. We know that $\nabla \cdot \vec{P} = -\rho_p$, which in integral form is written as

$$\oint_S \vec{P} \cdot d\vec{s} = -\int_V \rho_p \, dV$$

Because the field changes discontinuously at the surface we can reduce the size of the volume to an infinitesimal thickness around the interface in order to find the total amount of charge at the interface. In that case we must change the volume integral to a surface integral and the volume charge density to a surface charge density. The RHS of the equation then becomes

$$-\int_S \rho_{ps} \, ds$$

where ρ_{ps} is the polarization surface charge density. The left-hand side reduces to two surface integrals because the edges have infinitesimal size. Thus

$$\oint_S \vec{P} \cdot d\vec{s} = \int_{\text{S in region 0}} \vec{P} \cdot d\vec{s}_1 + \int_{\text{S in region 1}} \vec{P} \cdot d\vec{s}_2$$

Where $d\vec{s}_1$ and $d\vec{s}_2$ are surface vector elements which point away from the interface (in the case of 1 that would radially outward, and in the case of 0 that would be radially inward). So in writing this we were careful to consider the positive directions of flux through the surfaces. They have to be away from the interface. We note that $d\vec{s}_0 = -d\vec{s}_1$, and thus we can write

$$\oint_S \vec{P} \cdot d\vec{s} = \int_S \left(\vec{P}_1 - \vec{P}_0\right) \cdot d\vec{s}$$

Because the fields are perpendicular to the surface we can simplify to

$$\int_S P_{\rho 1} - P_{\rho 0} \, ds$$

Now combine the LHS and the RHS to get

$$\int_S P_{\rho 1} - P_{\rho 0} ds = \int_S \rho_{ps} ds$$

The integrands must be the same, so we get

$$P_{\rho 1} - P_{\rho 0} = -\rho_{ps}$$

Now, at the interface between the conductor and region 1 we have $P_{\rho 0} = 0$ and thus

$$P_{\rho 1} = -\rho_{ps}$$

$$\rho_{ps} = -P_{\rho 1}(\rho = a) = -\frac{\rho_l}{6\pi a}$$

(note the units work out OK as they should: linear charge density diviced by length equal surface charge density). At the interface between region 1 and region 2 we get

$$\begin{aligned}
\rho_{ps} &= - \left(P_{\rho 2}(\rho = r_1) - P_{\rho 1}(\rho = r_1) \right) \\
&= - \left(\frac{7\rho_l}{18\pi r_1} - \frac{\rho_l}{6\pi r_1} \right) \\
&= - \frac{\rho_l}{\pi r_1} \left(\frac{7}{18} - \frac{1}{6} \right) \\
&= - \frac{\rho_l}{\pi r_1} \left(\frac{7}{18} - \frac{3}{18} \right) \\
&= - \frac{\rho_l}{\pi r_1} \frac{4}{18} \\
&= - \frac{2\rho_l}{9\pi r_1}
\end{aligned}$$

(c) The polarization charge density (volume density) in region 2 is simply the negative of the divergence of the polarization vector. It is in cylindrical coordinates, and only the $\hat{\rho}$ component is non-zero. Thus we can write

$$\begin{aligned}
\rho_p &= - \nabla \cdot \vec{P} \\
&= - \frac{1}{\rho} \frac{\partial \rho P_\rho}{\partial \rho} \\
&= - \frac{1}{\rho} \frac{\partial}{\partial \rho} \frac{\rho_l}{6\pi} \\
&= 0
\end{aligned}$$