EE 333 Electricity and Magnetism, Fall 2009
Homework #6 solution

3.8.

(a) In characterizing materials according to their reactions to externally applied electric and magnetic field, we in general identified three different types of materials.

(i) Indicate these three types of materials and explain (in a few words) their basic characteristics.

(ii) Identify the induced charge and current sources as a result of the interaction of the external electric and magnetic fields with these materials.

(iii) Explain the impact of these new induced sources on Maxwell's equations.

(b) A spherical conductor of radius a is charged with a total positive charge Q. If the conductor is coated with two different dielectric materials of radii r_1 and r_2 as shown in Figure P3.8, determine

(i) The electric flux density $\vec{D}$, the electric field intensity $\vec{E}$, the polarization, $\vec{P}$, and the polarization charge density ρ_p in regions 1, 2, and 3.

(ii) The polarization surface charge density ρ_{ps} at the interface between regions 1 and 2 (i.e. at $r = r_1$).

(a) (i) The three type of materials are conducting materials, dielectric materials, and magnetic materials. In conducting materials a current will flow in response to a applied external electric field. In a dielectric material charged will not flow but will shift slightly in response to an external applied electric field. The shifted currents result in an electric field which points opposite the external field, thus reducing the electric field in the material. In a magnetic material internal dipole moments are rotated or created with the result that a magnetic field is created. This magnetic field can either point in the direction of the externally applied magnetic field or in the opposite direction.

(ii) In the conductor a current will flow in the direction of the electric field. In a dielectric charges are induced, with negative charges shifted in the direction of the electric field and positive charges shifted in the opposite direction of the electric field. This will cause a net charge in some locations: at surfaces, and in places where the applied field diverges. In a magnetic field the induced magnetic moments are described in terms of currents.

(iii) Maxwell's equations still hold, but instead of only considering the applied field, charges, and currents, we must consider all fields, charges, and currents. This is more complicated. To reduce the complication we invent the quantities $\vec{D}$ and $\vec{H}$, which only respond to the externally applied charges and currents, and not to any that are induced in materials. Once the problems is solved in terms of $\vec{D}$ and $\vec{H}$, we can then determine $\vec{E}$, and $\vec{B}$, which are the physical quantities.

(b) (i) Because of symmetry the fields must be radia. Use Gauss' law in integral form

$$\oint_S \vec{D} \cdot d\vec{s} = \int_V \rho \, dV$$

and let S be a spherical surface of radius $r > a$, and we get

$$4\pi r^2 D_r = Q$$

or

$$D_r = \frac{Q}{4\pi r^2}$$

Next, $\vec{E} = E_r \hat{r}$, and $E_r = \frac{D_r}{\epsilon_r \epsilon_0}$, and

$$E_r = \begin{cases} \frac{Q}{4\pi\epsilon_{1r}\epsilon_0 r^2} & a < r \leq r_1 \\ \frac{Q}{4\pi\epsilon_{2r}\epsilon_0 r^2} & r_1 < r \leq r_2 \\ \frac{Q}{4\pi\epsilon_0 r^2} & r_3 < r \end{cases}$$

Next, $\vec{P} = \epsilon_0 \chi_e \vec{E}$,

$$P_r = \epsilon_0 \chi_e E_r = \epsilon_0 \left(\epsilon_r - 1\right) E_r = \epsilon_0 \left(\epsilon_r - 1\right) \frac{D_r}{\epsilon_r \epsilon_0} = \left(1 - \frac{1}{\epsilon_r}\right) D_r$$

so

$$P_r = \begin{cases} \left(1 - \frac{1}{\epsilon_{1r}}\right) \frac{Q}{4\pi r^2} & a < r \leq r_1 \\ \left(1 - \frac{1}{\epsilon_{2r}}\right) \frac{Q}{4\pi r^2} & r_1 < r \leq r_2 \\ 0 & r_2 < r \end{cases}$$

To compute the polarization volume charge density we must compute the divergence, and the divergence of $\vec{P}$ is zero in all three regions. Therefore, $\rho_p = 0$ in all three regions.

(ii) The polarization surface charge density is found from Gauss' law for polarization charges which says that

$$\oint_S \vec{P} \cdot d\vec{s} = -\int_V \rho_p \, dV$$

and can be transformed to

$$\left(\vec{P}_2 - \vec{P}_1\right) \cdot \hat{n} = -\sigma_P$$

where $\hat{n}$ is a normal vector which points from region 1 into region 2, and $\vec{P}_2$ and $\vec{P}_1$ are the limits of $\vec{P}$ when the surface is appraoch from region 2 and region 1 respectively. At the interface between region 1 and region 2 we then have (choosing $\hat{n} = \hat{r}$)

$$\begin{aligned}\sigma_p &= -\left(P_{2r} - P_{1r}\right) \\ &= \left[\left(1 - \frac{1}{\epsilon_{1r}}\right) - \left(1 - \frac{1}{\epsilon_{2r}}\right)\right] \frac{Q}{4\pi r^2} \\ &= \left[\frac{1}{\epsilon_{2r}} - \frac{1}{\epsilon_{1r}}\right] \frac{Q}{4\pi r^2}\end{aligned}$$

We can then immediately by analog see that at the region 2 to 3 interface we must have

$$\sigma_p = \left[1 - \frac{1}{\epsilon_{2r}}\right] \frac{Q}{4\pi r^2}$$

3.9. The interface between regions 1 and 2 is charged with a surface charge density $\rho_s = 0.2\,\mathrm{C/m^2}$. Region 1 ($z > 0$) is air, whereas region 2 ($z < 0$) is a material with $\epsilon_2 = 2\epsilon_0$ and $\mu_2 = 3.1\mu_0$. If the electric flux density in region 1 is given by $\vec{D}_1 = 3\hat{x} + 4\sqrt{y}\hat{y} + 3\hat{z}$, and the magnetic field intensity in region 2 is

$$\vec{H}_2 = 4\hat{x} + 3y^2\hat{y} + 5\hat{z}$$

determine the electric flux density ($\vec{D}_2$) and the magnetic flux density ($\vec{B}_1$) at the interface between regions 1 and 2 in Figure P3.9.

For figuring $\vec{D}_2$ I am going to use the two boundary conditions,

$$\left(\vec{E}_2 - \vec{E}_1\right) \times \hat{n} = 0$$

and

$$\left(\vec{D}_1 - \vec{D}_2\right) \cdot \hat{n} = \sigma$$

where $\hat{n}$ goes from region 2 to region 1, and is parallel to the z-axis. So we can find D_{2z} from

$$D_{1z} - D_{2z} = \sigma$$

$$D_{2z} = D_{1z} - \sigma = 3 - 0.2 = 2.8\,\mathrm{C/m^2}$$

For the x and y components we see that

$$\vec{E}_{1\parallel} = \vec{E}_{2\parallel}$$

or

$$\frac{\vec{D}_{1\|}}{\epsilon_1} = \frac{\vec{D}_{2\|}}{\epsilon_2}$$

or

$$\vec{D}_{2\|} = \frac{\epsilon_2}{\epsilon_1}\vec{D}_{1\|}$$

And note that $\vec{D}_{1\|} = \hat{x}D_{1x} + \hat{y}D_{1y}$, and $\epsilon_2 = 2\epsilon_1$, so

$$\vec{D}_{2\|} = 2\left[3\hat{x} + 4\sqrt{y}\hat{y}\right] = 6\hat{x} + 8\sqrt{y}\hat{y}$$

Now we can combine, $\vec{D}_2 = \hat{z}D_{2z} + \vec{D}_{2\|}$ and get

$$\vec{D}_2 = 6\hat{x} + 8\sqrt{y}\hat{y} + 2.8\hat{z}\,\mathrm{C/m^2}$$

at the interface! Next we wish to find $\vec{B}_1$. They mention no surface current, so let's assumet that $\vec{J}_s = 0$, and we can then write

$$\left(\vec{B}_1 - \vec{B}_2\right)\cdot\hat{n} = 0$$

and

$$\left(\vec{H}_1 - \vec{H}_2\right)\times\hat{n} = 0$$

From the first equation we get that

$$B_{1z} = B_{2z} = \mu H_{2z} = 3.1\mu_0 \times 5 = 15.5\mu_0 = 15.5\times 4\pi\times 10^{-7} = 19.5\mu\mathrm{T}$$

For the x and y component we note that

$$\vec{H}_{1\|} = \vec{H}_{2\|}$$

or

$$\vec{B}_{1\|} = \mu_1\vec{H}_{2\|}$$

$$\begin{aligned}\vec{B}_{1\|} =& \mu_0\left(4\hat{x} + 3y^2\hat{y}\right)\\ =& 5.03\,\mu\mathrm{T}\hat{x} + 3.77\frac{\mu\mathrm{T}}{m^2}y^2\hat{y}\end{aligned}$$

Now we can combine terms and get

$$\vec{B}_1 = 5.03\,\mu\mathrm{T}\,\hat{x} + 3.77\,\frac{\mu\mathrm{T}}{m^2}y^2\,\hat{y} + 19.5\,\mu\mathrm{T}\,\hat{z}$$

at the interface!

3.15 (b). A plane wave is incident normal to the surface of sea water having the following constant $\mu_r = 1$, $\epsilon_r = 79$, and $\sigma = 3\,\Omega^{-1}\mathrm{m}^{-1}$. The electric field is

parallel to the surface and its magnitude is 10 V/m just inside the surface of the water. At what depth would it be possble for a submarine to receive a signal if the sub's receiver requires a field intensity of 10 μV/m? Make your calculations at the following two frequencies:

(i) 20 kHz. (Can displacement current be neglected?)

(ii) 20 GHz. (Can the conduction current be neglected?)

(i) Use the spatial decay constant

$$\alpha = \frac{\omega\sqrt{\mu\epsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1}$$

The question of whether the displacement current can be ignored relates to Ampere's law. We have

$$\nabla \times \vec{H} = \sigma\vec{E} + \epsilon\frac{\partial\vec{E}}{\partial t}$$

which can be re-written, assumingt that $\vec{E}$ has the time-variation $e^{j\omega t}$ as

$$\nabla \times \vec{H} = \sigma\vec{E} + j\omega\epsilon\vec{E}$$

The displacement current can thus be ignored if

$$\omega\epsilon \ll \sigma$$

so let's determine

$$\frac{\sigma}{\omega\epsilon} = \frac{3}{2\times\pi\times 20\times 10^3\times 8.854\times 10^{-12}} = 2.696\times 10^6$$

It is therefore safe to ingore the displacement current and simplify to

$$\alpha = \frac{\omega\sqrt{\mu\epsilon}}{\sqrt{2}}\sqrt{\frac{\sigma}{\omega\epsilon}} = \sqrt{\frac{\omega\mu\sigma}{2}}$$

Inserting known values we get

$$\begin{aligned}\alpha =& \sqrt{\frac{2\times 20\times 10^3\times 4\times\pi\times 10^{-7}\times 3}{2}}\\ =& 0.487\,\mathrm{m}^{-1}\end{aligned}$$

The depth, d, at which the signal can be detected is then determined from

$$E_{\text{detect}} = E_{\text{initial}} e^{-\alpha\, d}$$

or

$$d = \frac{1}{\alpha} \ln \frac{E_{\text{initial}}}{E_{\text{detect}}}$$

Inserting values we get

$$d = \frac{1}{0.487} \ln \frac{10}{10 \times 10^{-6}} = 28.4\,\text{m}$$

(ii) This time the question is whether the conduction current can be ignored. Again we evaluate

$$\frac{\sigma}{\omega\epsilon} = \frac{3}{2 \times \pi \times 20 \times 10^9 \times 79 \times 8.854 \times 10^{-12}} = 0.034$$

which is small. So we can Taylor expand

$$\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} = 1 + \frac{1}{2}\left(\frac{\sigma}{\omega\epsilon}\right)^2$$

and thus

$$\sqrt{\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1} = \sqrt{\frac{1}{2}\left(\frac{\sigma}{\omega\epsilon}\right)^2} = \frac{\sigma}{\sqrt{2}\omega\epsilon}$$

Now the expression for α simplifies to

$$\alpha = \frac{\omega\sqrt{\mu\epsilon}}{\sqrt{2}} \frac{\sigma}{\sqrt{2}\omega\epsilon} = \sqrt{\frac{\mu}{\epsilon}} \frac{\sigma}{2}$$

Inserting known values we get

$$\alpha = \sqrt{\frac{4 \times \pi \times 10^{-7}}{79 \times 8.854 \times 10^{-12}}} \times \frac{3}{2} = 63.6\,\text{m}^{-1}$$

and the penetration depth is

$$d = \frac{1}{63.6} \ln 10^6 = 0.22\,\text{m}$$

3.17. A high-voltage wire of radius a is insulated with an insulation coating of radius b and dielectric constant $\epsilon = \epsilon_r \epsilon_0$. The insulated high-voltage wire is suspended at the center of a grounded pipe of radius c. The geometry of the suspended cable is shown in Figure P3.17. When a high-votage V is applied between the center wire and the grounded pipe, a charge ρ_s (per unit area) was added to the center of the conductor.

(a) Determine the electric flux density $\vec{D}$ inside the insulation (region 1) and in the air (region 2).

(b) Determine the electric field intensity $\vec{E}$ and the polarization $\vec{P}$ in regions 1 and 2.

(c) Determine the free charge density ρ_s at the surface of the grounded pipe of radius c.

(d) Determine the induced surface polarization charge at the interface between regions 1 and 2, that is at $\rho = b$.

(e) Plot the electric field $\vec{E}$ as a function of ρ for $a < \rho < c$.

(f) Repeat part e for the case in which we replace region 1 by air and place the dielectric material $\epsilon = \epsilon_0 \epsilon_r$ in region 2.

(g) As a result of the plots in parts e and f, which case is better from the insulation viewpoing?

(a) We use Gauss' law for electric fields,

$$\oint_S \vec{D} \cdot d\vec{s} = \int_V \rho \, dV$$

Apply it to a length, L of pipe, realize that D is radial, and we get

$$2\pi \rho L D_\rho = 2\pi a L \rho_s$$

or

$$D_\rho = \frac{a}{\rho} \rho_s$$

which is valid in both region 1 and 2.

(b) The electric field is

$$\vec{E} = \frac{\vec{D}}{\epsilon}$$

so we get

$$E_r = \begin{cases} \frac{a}{\rho}\frac{\rho_s}{\epsilon_r\epsilon_0} & a < \rho < b \\ \frac{a}{\rho}\frac{\rho_s}{\epsilon_0} & b < \rho < c \end{cases}$$

The polarziation is $\vec{P} = \epsilon_0\chi_e\vec{E} = \epsilon_0\left(1-\epsilon_r\right)\vec{E} = \left(1-\epsilon_r\right)\frac{\vec{D}}{\epsilon_r} = \left(1-\frac{1}{\epsilon_r}\right)\vec{D}$, such that

$$P_r = \begin{cases} \left(1-\frac{1}{\epsilon_r}\right)\frac{a}{\rho}\rho_s & a < \rho < b \\ 0 & b < \rho < c \end{cases}$$

(c) The field outside the grounded pipe should be zero, so the total amount of charge inside the pipe should be zero, so

$$2\pi a\rho_{sa} = 2\pi c\rho_{sc}$$

or

$$\rho_{sc} = \rho_{sa}\frac{a}{c}$$

(d) We have for polarization fields

$$\left(\vec{P}_2 - \vec{P}_1\right)\cdot\hat{n} = -\rho_{sp}$$

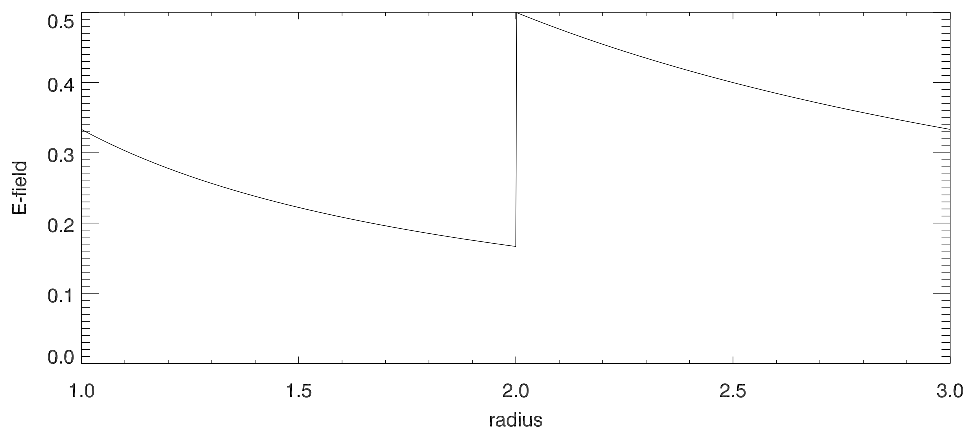
where $\hat{n}$ points in the direction from region 1 into region 2. Choose $\hat{n} = r\hat{h}o$, and we get

$$P_{2r} - P_{1r} = -\sigma_{ps}$$

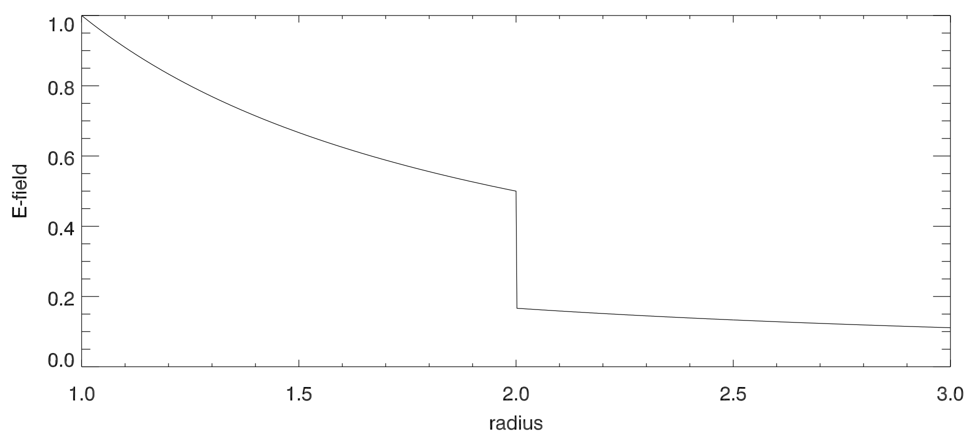
or

$$\sigma_{ps} = P_{1r} - P_{2r} = \left(1-\frac{1}{\epsilon_r}\right)\frac{a}{b}\rho_s$$

(e) To do this I pick some numbers: $\epsilon_r = 3$, $a = 1$, $b = 2$, $c = 3$, $\epsilon_0 = 1$, and $\sigma_s = 1$.



(f) To do this I simply move the ϵ_r from the upper equation tot he lower equation in the formula for E_ρ, and get



(g) If the purpose of the insulator is to reduce the maximum electric field, then placing the insulator at the center is best. If on the other hand the purpose of the insulator is to create the smallest electric field at the inside surface of the pipe it is better to place the insulator close to the pipe.