EE 333 Electricity and Magnetism, Fall 2009
Homework #7 solution

4.4. Consider the linear quadrupole shown in Figure P4.4. It basically consists of two dipoles superposed along the z axis. Determine the potential P at a far distance r (i.e. $r \gg d$) from the charges and the electric field at that point. Use the approximations

$$\frac{1}{r_1} \approx \frac{1}{r}\left(1 + \frac{d}{r}\cos\theta\right)$$
$$\frac{1}{r_2} \approx \frac{1}{r}\left(1 - \frac{d}{r}\cos\theta\right)$$

The potential is

$$\begin{aligned}\Phi =& \sum_i \frac{Q_i}{4\pi\epsilon_0 R_i}\\ =& \frac{-Q}{4\pi\epsilon_0 r_1} + \frac{2Q}{4\pi\epsilon_0 r} + \frac{-Q}{4\pi\epsilon_0 r_2}\\ =& \frac{Q}{4\pi\epsilon_0}\left(\frac{2}{r} - \frac{1}{r_1} - \frac{1}{r_2}\right)\end{aligned}$$

Now using the approximations we get

$$\begin{aligned}\Phi =& \frac{Q}{4\pi\epsilon_0}\left(\frac{2}{r} - \frac{1}{r}\left(1 + \frac{d}{r}\cos\theta\right) - \frac{1}{r}\left(1 - \frac{d}{r}\cos\theta\right)\right)\\ =& 0\end{aligned}$$

In order to get a better approximation we would need to expand to higher degree.

4.5. Following a procedure similar to the one illustrated in examples 4.7 and 4.8, determine the capacitance of two concentric spherical conductors of radii a and b when a spherical dielectric shell of thickness d is placed concentrically between the conductors as shown in Figure P4.5a. The dielectric shell has an inner radius c and dielectric constant $\epsilon = \epsilon_0\epsilon_s$. Show that the total capacitance is the sum of three series capacitances each with a homogeneous dielectric layer as shown in Figure P4.5b.

We do this by applying charge Q to the inner conductor and charge $-Q$ to the outer conductor and then computing the potential difference between them. Using Gauss' law and symmetry we get we get

$$4\pi r^2 D_r = Q$$

$$D_r = \frac{Q}{4\pi r^2}$$

And the electric field is then

$$E_r = \begin{cases} \frac{Q}{4\pi\epsilon_0 r^2} & a < r < c \\ \frac{Q}{4\pi\epsilon_0\epsilon_r r^2} & c < r < c + d \\ \frac{Q}{4\pi\epsilon_0 r^2} & c + d < r < b \end{cases}$$

Next, we integrate the electric field

$$\begin{aligned}
V &= \int_a^b \vec{E} \cdot d\vec{l} \\
&= \int_a^b E_r \, dr \\
&= \int_a^c \frac{Q}{4\pi\epsilon_0 r^2} dr + \int_c^{c+d} \frac{Q}{4\pi\epsilon_0\epsilon_r r^2} dr + \int_{c+d}^b \frac{Q}{4\pi\epsilon_0 r^2} dr \\
&= \left[\frac{-Q}{4\pi\epsilon_0 r}\right]_a^c + \left[\frac{-Q}{4\pi\epsilon_0\epsilon_r r}\right]_c^{c+d} + \left[\frac{-Q}{4\pi\epsilon_0 r}\right]_{c+d}^b \\
&= \frac{Q}{4\pi\epsilon_0}\left[\left(\frac{1}{a} - \frac{1}{c}\right) + \frac{1}{\epsilon_r}\left(\frac{1}{c} - \frac{1}{c+d}\right) + \left(\frac{1}{c+d} - \frac{1}{b}\right)\right]
\end{aligned}$$

and the capacitacen is then

$$C = \frac{Q}{V} = \frac{4\pi\epsilon_0}{\left(\frac{1}{a} - \frac{1}{c}\right) + \frac{1}{\epsilon_r}\left(\frac{1}{c} - \frac{1}{c+d}\right) + \left(\frac{1}{c+d} - \frac{1}{b}\right)}$$

From the voltage integral above we can fairly quickly see that if we had a conductor at c, with charge $-Q$ the voltage drop would be

$$V_1 = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{c}\right)$$

resulting in a capacitance

$$C_1 = \frac{4\pi\epsilon_0}{\frac{1}{a} - \frac{1}{c}}$$

and similar for $[c; c + d]$, we get

$$C_2 = \frac{4\pi\epsilon_0\epsilon_r}{\frac{1}{c} - \frac{1}{c+d}}$$

and similar for $[c + d; b]$, we get

$$C_3 = \frac{4\pi\epsilon_0}{\frac{1}{c+d} - \frac{1}{b}}$$

Now replacing these into the denominator of the expression for C, we get

$$
\begin{aligned}
C =& \frac{4\pi\epsilon_0}{\frac{4\pi\epsilon_0}{C_1} + \frac{1}{\epsilon_r}\frac{4\pi\epsilon_r}{C_2} + \frac{4\pi\epsilon_0}{C_3}} \\
=& \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}}
\end{aligned}
$$

which is the series combination of C_1, C_2, and C_3.