EE 333 Electricity and Magnetism, Fall 2009
Homework #9 solution

4.10. The two infinite conducting cones $\boldsymbol{\theta = \theta_1}$, and $\boldsymbol{\theta = \theta_2}$ are maintained at the two potentials $\boldsymbol{\Phi_1 = 100\,\mathrm{V}}$, and $\boldsymbol{\Phi_2 = 0\,\mathrm{V}}$, respectively, as shown in Figure P4.10.

(a) Use Laplace's equation in the spherical coordinates to solve for the potential variation between the two cones.

(b) Calculate the electric field vector in the region between the two cones and the charge density on each conductor.

(a) In spherical coordinates we have

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

Now, first we realize that because of rotational symmetry around the z-axis ($\theta = 0$), the potential will not depend on ϕ, so the last term in the Laplace equation is zero. Next we realize that we can scale the cones up and down in size, and because they extend to infinite, the system is unchanged. Therefore, the potential also does not depend on r. We are then left only with the θ-variation:

$$\nabla^2\Phi = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) = 0$$

or

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) = 0$$

$$\sin\theta\frac{\partial\Phi}{\partial\theta} = \alpha$$

$$\frac{\partial\Phi}{\partial\theta} = \frac{\alpha}{\sin\theta}$$

With the help of an integral table we get

$$\Phi = \alpha\ln\left|\tan\frac{\theta}{2}\right| + \beta$$

Since $\theta \in [0; 180°]$, we can remove the absolute value sign to get

$$\Phi(r,\theta,\phi) = \alpha\ln\left(\tan\frac{\theta}{2}\right) + \beta$$

Next we apply the boundary conditions:

$$\Phi\left(r,\theta_1,\phi\right)=100\,\mathrm{V}$$

and

$$\Phi\left(r,\theta_2,\phi\right)=0\,\mathrm{V}$$

We see that

$$\beta=-\alpha\ln\left(\tan\frac{\theta_2}{2}\right)$$

and then

$$\alpha\ln\left(\tan\frac{\theta_1}{2}\right)-\alpha\ln\left(\tan\frac{\theta_2}{2}\right)=100\,\mathrm{V}$$

or

$$\begin{aligned}\alpha=&\frac{100\,\mathrm{V}}{\ln\tan\frac{\theta_1}{2}-\ln\tan\frac{\theta_2}{2}}\\=&\frac{100\,\mathrm{V}}{\ln\frac{\tan\frac{\theta_1}{2}}{\tan\frac{\theta_2}{2}}}\end{aligned}$$

(Note, $\alpha<0$)

(b) The electric field is the gradient of the potential. In spherical coordinates we have

$$\vec{E}=-\nabla\Phi=-\frac{\partial\Phi}{\partial r}\hat{r}-\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\theta}-\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\hat{\phi}$$

Because Φ only depends on θ it simplifies to

$$\begin{aligned}\vec{E}=&-\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\theta}\\=&-\frac{1}{r}\frac{\partial}{\partial\theta}\left(\alpha\ln\tan\frac{\theta}{2}+\beta\right)\hat{\theta}\\=&-\frac{1}{r}\frac{\alpha}{\sin\theta}\hat{\theta}\end{aligned}$$

The surface charge density on the upper conductor is

$$\sigma_1 = \epsilon_0 E_{1\theta} = -\frac{1}{r}\frac{\alpha}{\sin\theta_1}$$

(it is positive)

The surface charge density on the lower conductor is

$$\sigma_2 = -\epsilon_0 E_{2\theta} = \frac{1}{r}\frac{\alpha}{\sin\theta_2}$$

(it is negative)

4.11. Consider the two parallel plates shown in Figure P4.11. The region between the parallel plates is filled with a nonunifom charge distribution of density $\rho(y) = \sigma_o y$, where σ_o is a constant. Solve Poisson's equation in the region between the parallel plates to show that the potential distribution $\Phi(y)$ is given by

$$\Phi(y) = \frac{V}{d}y + \frac{\sigma_o}{6\epsilon_o}\left(yd^2 - y^3\right)$$

Poisson's equation in cartesian coordinates looks like this:

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = -\frac{\rho}{\epsilon_0}$$

Because of symmetry only the y-derivative is non-zero, so

$$\frac{\partial^2\Phi}{\partial y^2} = -\frac{\sigma_o y}{\epsilon_0}$$

and thus

$$\frac{\partial\Phi}{\partial y} = -\frac{\sigma_o}{2\epsilon_0}y^2 + \alpha$$

$$\Phi = -\frac{\sigma_o}{6\epsilon_0}y^3 + \alpha y + \beta$$

The boundary conditions are

$$\Phi(0) = 0 \qquad \Phi(d) = V$$

and thus

$$\beta = 0$$

and

$$V = -\frac{\sigma_o d^3}{6\epsilon_0} + \alpha d$$

$$\alpha = \frac{V}{d} + \frac{\sigma_o d^2}{6\epsilon_0}$$

Inserting we get

$$\begin{aligned}\Phi =& -\frac{\sigma_o y^3}{6\epsilon_0} + \left(\frac{V}{d} + \frac{\sigma_o d^2}{6\epsilon_0}\right) y \\ =& \frac{V}{d} y - \frac{\sigma_o y^3}{6\epsilon_0} + \frac{\sigma_o d^2}{6\epsilon_0} y \\ =& \frac{V}{d} y + \frac{\sigma_o}{6\epsilon_0}\left(d^2 y - y^3\right)\end{aligned}$$

4.12. Use the expression of the potential distribution in problem 11 to obtain an expression for the electric field between the two plates. Show that the charge density at the lower plate is given by

$$\boldsymbol{\rho_s = -\epsilon_0 \left(\frac{V}{d} + \frac{\sigma_o}{6\epsilon_0} d^2\right)}$$

whereas the charge density at the upper plate is given by

$$\boldsymbol{\rho_s = \epsilon_0 \left(\frac{V}{d} - \frac{\sigma_o}{3\epsilon_0} d^2\right)}$$

The capacitance is defined as $\boldsymbol{C = \frac{Q}{V}}$ and because in this case there are two different values of $\boldsymbol{Q}$ on the lower and upper plates for the same potential difference $\boldsymbol{V}$, there is no unique value for the capacitance under these circumstances.
Compute the electric field as

$$\vec{E} = -\nabla\Phi$$

Since the potential only varies with y, the electric field points in the $\hat{y}$ direction, and

$$\begin{aligned}\vec{E} =& -\hat{y}\frac{\partial \Phi}{\partial y} \\ =& -\hat{y}\frac{\partial}{\partial y}\left[\frac{V}{d} y + \frac{\sigma_o}{6\epsilon_0}\left(y d^2 - y^3\right)\right] \\ =& -\hat{y}\left[\frac{V}{d} + \frac{\sigma_o}{6\epsilon_0}\left(d^2 - 3y^2\right)\right]\end{aligned}$$

This is the field pointing in the positive $\hat{y}$ direction, so the charge density on the lower conductor is

$$
\begin{aligned}
\rho_s =& \epsilon_0 E_y(y=0) \\
=& -\epsilon_0 \frac{V}{d} - \frac{\sigma_o}{6}\left(d^2 - 0\right) \\
=& -\epsilon_0 \frac{V}{d} - \frac{\sigma_o d^2}{6} \\
=& -\epsilon_0 \left(\frac{V}{d} + \frac{\sigma_o d^2}{6\epsilon_0}\right)
\end{aligned}
$$

and the charge density on the upper conductor is

$$
\begin{aligned}
\rho_s =& -\epsilon_0 E_y(y=d) \\
=& \frac{V\epsilon_0}{d} + \frac{\sigma_o}{6}\left(d^2 - 3d^2\right) \\
=& \frac{V\epsilon_0}{d} - \frac{2\sigma_o d^2}{6} \\
=& \epsilon_0 \left(\frac{V}{d} - \frac{\sigma_o d^2}{3}\right)
\end{aligned}
$$