EE 333 Electricity and Magnetism, Fall 2009
Homework #11 solution

4.24. At the interface between two magnetic materials shown in Fig P4.24, a surface current density $\boldsymbol{J_S = 0.1\,\hat{y}}$ is flowing. The magnetic field intensity $\boldsymbol{\vec{H}_2}$ in region 2 is given by $\boldsymbol{\vec{H}_2 = 3\hat{x} + 9\hat{z}}$. Determine the magnetic flux $\boldsymbol{\vec{B}_1}$ and $\boldsymbol{\vec{B}_2}$ in regions 1 and 2, respectively.

We have two boundary equations,

$$\hat{n} \cdot \left(\vec{B}_1 - \vec{B}_2\right) = 0 \qquad \hat{n} \times \left(\vec{H}_1 - \vec{H}_2\right) = \vec{J}_s$$

From the first equation we get simply that $B_{1z} = B_{2z}$. And, $B_{1z} = B_{2z} = 3\mu_0 H_{2z} = 27\mu_0$. From the second equation we see that (since $\hat{n} = \hat{z}$ that

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ H_{1x} - H_{2x} & H_{1y} - H_{2y} & H_{1z} - H_{2z} \end{vmatrix} = \hat{y} J_{sy}$$

Which reduces to

$$H_{1y} - H_{2y} = 0 \qquad H_{1x} - H_{2x} = J_{sy}$$

So $H_{1y} = H_{2y} = 0$, and $H_{1x} = H_{2x} + J_{sy} = 3 + 0.1 = 3.1$. Now compute the magnetic field components, $B_{1x} = 5\mu_0 H_{1x} = 15.5\mu_0$, and $B_{1y} = 0$. The magnetic field in region 1 is then

$$\vec{B}_1 = 15.5\mu_0\hat{x} + 27\mu_0\hat{z}$$

In region 2 we have $B_{2x} = 3\mu_0 H_{2x} = 9\mu_0$, and $B_{2y} = 0$, so overall we get

$$\vec{B}_2 = 9\mu_0\hat{x} + 27\mu_0\hat{x}$$

4.25. Consider the problem of determining the magnetic vector potential $\boldsymbol{\vec{A}}$ inside and outside an infinite circular cylindrical solenoid of radius $\boldsymbol{a}$. The solenoid has $\boldsymbol{N}$ turns per unit length and the current in the winding is $\boldsymbol{I}$.

(a) **Use the curl relation between $\boldsymbol{\vec{A}}$ and the magnetic field $\boldsymbol{\vec{B} = \nabla \times \vec{A}}$ and Stokes' theorem to show that**

$$\oint_c \vec{A} \cdot d\vec{l} = \int_s \vec{B} \cdot d\vec{s}$$

where $\boldsymbol{s}$ is the area encircled by $\boldsymbol{c}$.

(b) **Based on symmetry considerations, select suitable contours for $\boldsymbol{\vec{A}}$ inside and outside the solenoid to show that**

$$\vec{A} = \begin{cases} \frac{\mu_0 N I \rho}{2}\hat{\phi} & \rho < a \\ \frac{\mu_0 N I a^2}{2\rho}\hat{\phi} & a < \rho \end{cases}$$

(a) Stokes' theorem says that

$$\oint_c \vec{F} \cdot \vec{l} = \int_S \nabla \times \vec{F} \cdot d\vec{s}$$

If we set $\vec{F} = \vec{A}$ we get

$$\oint_c \vec{A} \cdot \vec{l} = \int_s \nabla \times \vec{A} \cdot d\vec{s} = \int_s \vec{B} \cdot d\vec{s}$$

QED

(b) To do this problem we first need to determine the magnetic field. From symmetry considerations we see that at each radial distance from the center of the solenoid, ρ, the magnetic field must point along the z-axis independent of the azimuthal angle. This is due to the rotational symmetry of the solenoid.

$$\oint_l \vec{B} \cdot d\vec{l} = \mu_0 \int_s \vec{J} \cdot d\vec{s}$$

with any rectangular contour that extends from beyond one side of the solenoid to beyond the opposite side of the solenoid. Only the parts of the contour along the axis contribute, and since those contributions must be opposite (yet same field), and the total current through the contour is zero, the field outside the solenoid must be zero.

Next consider another rectangular which extends from inside the solenoid to outside it. It extends distance L along the axis of the solenoid. Only the contour path along the axis of the solenoid, inside the solenoid, has non-zero magnetic field contribution. We then get

$$LB_z = \mu_0 LNI$$

or

$$B_z = \mu_0 NI$$

The magnetic field of a solenoid is then

$$\vec{B} = \begin{cases} \hat{z}\mu_0 NI & \text{inside} \\ 0 & \text{outside} \end{cases}$$

Now we are ready to compute $\vec{A}$. Pick a contour which is circular and goes in the right-hand direction around the z-axis at a distance ρ. Note that since $\vec{B} = \hat{z}B_z$, $\vec{A} = \hat{\phi}A_\phi$. Inside the solenoid we then have

$$2\pi\rho A_\phi = \pi\rho^2 B_z = \pi\rho^2 \mu_0 NI$$

or

$$A_\phi = \frac{\mu_0 \rho N I}{2}$$

whereas outside the solenoid we have

$$2\pi\rho A_\phi = \pi a^2 B_z = \pi a^2 \mu_0 N I$$

or

$$A_\phi = \frac{\mu_0 a^2 N I}{2\rho}$$

Putting it together we get

$$\vec{A} = \begin{cases} \hat{\phi}\frac{\mu_0 \rho N I}{2} & \rho \leq a \\ \hat{\phi}\frac{\mu_0 a^2 N I}{2\rho} & a < \rho \end{cases}$$

4.29. Use the result of equation 4.99a for the magnetic flux at a far point from a circular current loop to determine approximately the mutual inductance between two thin coaxial circular rings of radii a and b. Assume that the distance d between the two rings is much larger than a and b.
The magnetic field from a small current loop a carrying current I is

$$\vec{B} = \frac{\mu_0 I a^2}{4r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)$$

On the axis it reduces to

$$B_z = \frac{\mu_0 I a^2}{2r^3}$$

The amount of flux through a small coaxial loop of radius b at distance d is then

$$\psi = \pi b^2 B_z = \frac{\mu_0 \pi I a^2 b^2}{2d^3}$$

The inductance is then

$$L = \frac{\psi}{I} = \frac{\mu_0 \pi a^2 b^2}{2d^3}$$

Note that it is symmetric. I.e. same result whether a or b is generating the field with current I.
7.2. Consider the following voltage and current distributions:

$$\boldsymbol{v(z,t) = v_o \cos\beta\,(z - ut) \qquad i(z,t) = \frac{v_o}{Z_0}\cos\beta\,(z - ut)}$$

where β is a constant, $u = \frac{1}{\sqrt{lc}}$ and $Z_o = \sqrt{\frac{l}{c}}$. By direct substitution, verify that $v(z,t)$ and $i(z,t)$ satisty the transmission line equation 7.10 to 7.13.
The transmission line equations that we are to verify are

$$\begin{aligned}
-\frac{dv}{dz} =& l\frac{di}{dt} \\
-\frac{di}{dz} =& c\frac{dv}{dt} \\
\frac{d^2v}{dz^2} =& lc\frac{d^2v}{dt^2} \\
\frac{d^2i}{dz^2} =& lc\frac{d^2i}{dt^2}
\end{aligned}$$

Check the first equation

$$\begin{aligned}
v_o\beta\sin\beta\left(z-ut\right) =& l\beta u\frac{v_o}{Z_0}\sin\beta\left(z-ut\right) \\
1 =& \frac{lu}{Z_0} \\
1 =& \frac{l\frac{1}{\sqrt{lc}}}{\sqrt{\frac{l}{c}}} \\
1 =& \frac{\sqrt{\frac{l}{c}}}{\sqrt{\frac{l}{c}}} \\
1 =& 1
\end{aligned}$$

Check the second equation

$$\begin{aligned}
\frac{v_o}{Z_0}\beta\sin\beta\left(z-ut\right) =& cv_0\beta u\sin\beta\left(z-ut\right) \\
\frac{1}{Z_0} =& lu \\
\sqrt{\frac{c}{l}} =& \frac{c}{\sqrt{lc}} \\
\sqrt{\frac{c}{l}} =& \sqrt{\frac{c}{l}} \\
1 =& 1
\end{aligned}$$

Check the 3rd equation

$$
\begin{aligned}
v_o\beta^2 \cos\beta\,(z-ut) =& v_o lc\beta^2 u^2 \cos(z-ut) \\
1 =& lcu^2 \\
1 =& \frac{lc}{\left(\sqrt{lc}\right)^2} \\
1 =& 1
\end{aligned}
$$

Check the 4th equation

$$
\begin{aligned}
\frac{v_o}{Z_0}\beta^2 \cos\beta\,(z-ut) =& lc\frac{v_o}{Z_0}\beta^2 u^2 \cos\beta\,(z-ut) \\
1 =& lcu^2 \\
1 =& \frac{lc}{\left(\sqrt{lc}\right)^2} \\
1 =& 1
\end{aligned}
$$