EE 333 Electricity and Magnetism, Fall 2009
Homework #12 solution

7.4. Given the lossless transmission line in Figure P7.4, if it takes T seconds for a traveling wave to move from the sending to the receiving end of the line:

1. **Plot the voltage at the receiving end as a function of time up to $t = 9T$. Write the voltage value on the plot for each time interval.**
2. **Plot the sending-end current up to $t = 9T$ and write the current on the plot for each time interval.**
3. **What are the final values of the load voltage and current?**

1. The initial voltage wave has amplitude

$$v_0^+ = V_G \frac{Z_0}{R_G + Z_0} = 9 \times \frac{75}{75 + 150} = 3\,\mathrm{V}$$

The reflection coefficient at the receiving end is

$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} = \frac{50 - 75}{50 + 75} = -\frac{1}{5}$$

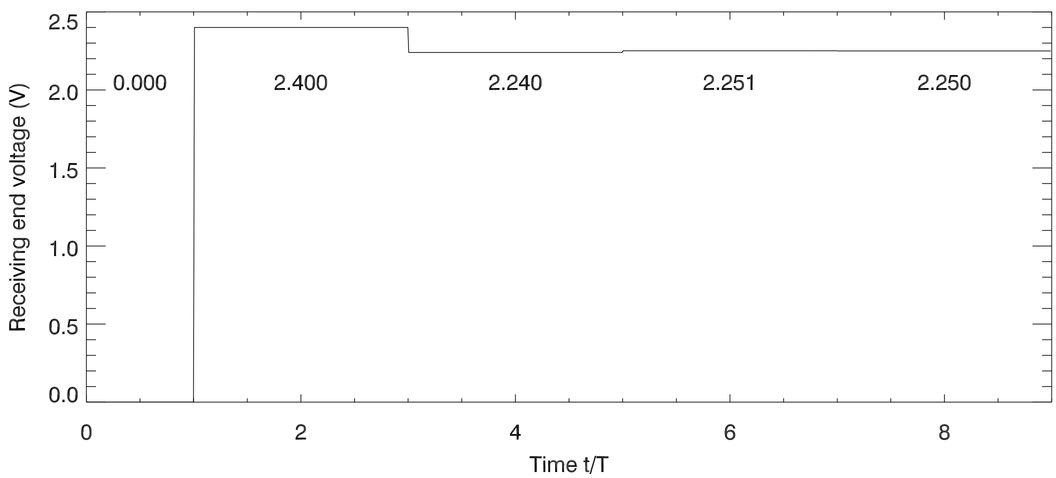
The reflection coefficient at the sending end is

$$\Gamma_G = \frac{R_G - Z_0}{R_G + Z_0} = \frac{150 - 75}{150 + 75} = \frac{1}{3}$$

At each time $t = 0, T, 2T, ...$ a wave is launched. I will label those v_0^+, v_1^-, v_2^+, etc. I already computed v_o^+ above. Then,

$$\begin{aligned}
v_0^+ &= 3\,\mathrm{V} \\
v_1^- &= \Gamma_L v_o^+ = -\frac{3}{5}\,\mathrm{V} = -0.6\,\mathrm{V} \\
v_2^+ &= \Gamma_G v_1^- = -\frac{1}{5}\,\mathrm{V} = -0.2\,\mathrm{V} \\
v_3^- &= \Gamma_L v_2^+ = \frac{1}{25}\,\mathrm{V} = 0.04\,\mathrm{V} \\
v_4^+ &= \Gamma_G v_3^- = \frac{1}{75}\,\mathrm{V} = 0.0133\,\mathrm{V} \\
v_5^- &= \Gamma_L v_4^+ = -\frac{1}{375}\,\mathrm{V} = 0.00266\,\mathrm{V} \\
v_6^+ &= \Gamma_G v_5^- = -\frac{1}{1125}\,\mathrm{V} = 0.000888\,\mathrm{V} \\
v_7^- &= \Gamma_L v_6^+ = \frac{1}{5625}\,\mathrm{V} = 0.000177\,\mathrm{V}
\end{aligned}$$

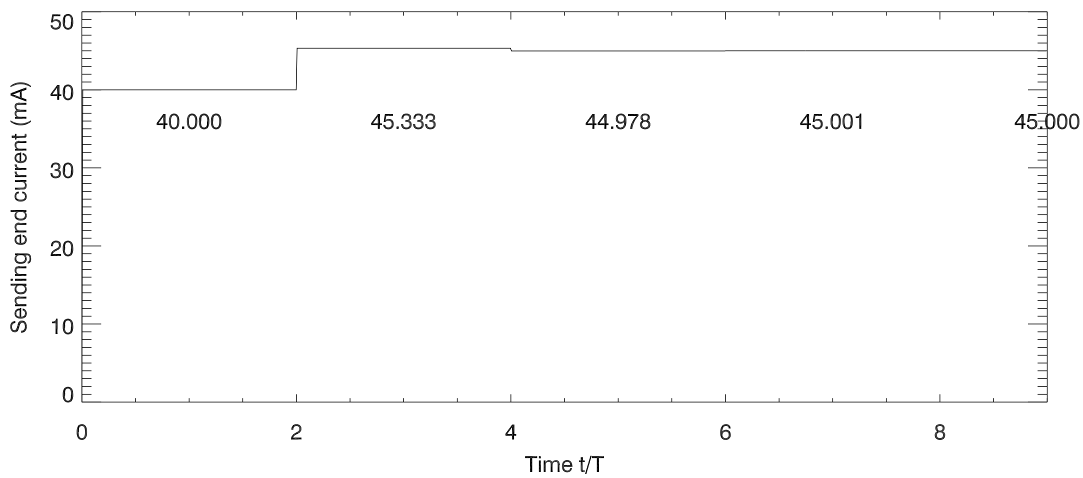
Now, up to time T, the voltage is $v_L = 0\,\text{V}$. For $T < t < 3T$, the voltage is $v_L = v_0^+ + v_1^-$. For $3T < t < 5T$, the voltage is $v_L = v_0^+ + v_1^- + v_2^+ v_3^-$. For $5T < t < 7T$, the voltage is $v_L = v_0^+ + v_1^- + v_2^+ + v_3^- + v_4^+ + v_5^-$. For $7T < t < 9T$ the voltage is $v_L = v_0^+ + v_1^- + v_2^+ + v_3^- + v_4^+ + v_5^- + v_6^+ + v_7^-$.



2. First compute the currents. For forward traveling waves, it is the voltage divided by the line impedance. For reverse traveling waves it is the negative of that, so we get

$$
\begin{aligned}
i_0^+ &= \frac{3}{75} = 40.0\,\text{mA} \\
i_1^- &= \frac{3}{375} = 8.00\,\text{mA} \\
i_2^+ &= -\frac{1}{375} = -2.67\,\text{mA} \\
i_3^- &= -\frac{1}{1875} = -0.533\,\text{mA} \\
i_4^+ &= \frac{1}{5625} = 0.178\,\text{mA} \\
i_5^- &= \frac{1}{28125} = 35.6\,\mu\text{A} \\
i_6^+ &= -\frac{1}{84375} = -11.9\,\mu\text{A} \\
i_7^- &= -\frac{1}{421875} = -2.37\,\mu\text{A} \\
i_8^+ &= \frac{1}{1265625} = 0.79\,\mu\text{A}
\end{aligned}
$$

Then, for $0 < t < 2T$, $i_G = i_0^+$, for $2T < t < 4T$, $i_G = i_0^+ + i_1^- + i_2^+$, for $4T < t < 6T$, $i_G = i_0^+ + i_1^- + i_2^+ + i_3^- + i_4^+$, for $6T < t < 8T$, $i_G = i_0^+ + i_1^- + i_2^+ + i_3^- + i_4^+ + i_5^- + i_6^+$, for $8T < t < 10T$, $i_G = i_0^+ + i_1^- + i_2^+ + i_3^- + i_4^+ + i_5^- + i_6^+ + i_7^- + i_8^+$



3. The final values for load voltage and current are those we expect from considering the transmission line a short,

$$v_L(t \to \infty) = \frac{R_L}{R_L + R_G} V_G = \frac{50}{50 + 150} \times 9 = 2.25\,\mathrm{V}$$

$$i_L(t \to \infty) = \frac{V_G}{R_L + R_G} = \frac{9}{50 + 150} = 45\,\mathrm{mA}$$

7.5. For the transmission line shown in Figure P7.5, obtain equations describing the variation of the load current with time. Plot this current variation as a function of time and explain physically the reason for the occurrence of such variation.

We begin by noting that at the load we have

$$v_T = v^+ + v^- \qquad i_T = i^+ + i^- = \frac{v^+}{Z_0} - \frac{v^-}{Z_0} \qquad Z_0 i_T = v^+ - v^-$$

Add the last and first equation and we get

$$v_T + Z_0 i_T = 2v^+$$

and we are asked to determine i_T. For an inductor we have the relationship between current and voltage which is

$$v_T = L\frac{di_T}{dt}$$

Substituting it in the above equation we get

$$L\frac{di_T}{dt} + Z_0 i_T = 2v^+$$

We look for a solution after time $t = 0$ when the pulse v^+ arrives at the inductor. The solution is

$$i_T = Ae^{-\frac{Z_0 t}{L}} + B$$

Inserting we get

$$L\frac{Z_0}{L}Ae^{-\frac{Z_0 t}{L}} + Z_0 Ae^{-\frac{Z_0 t}{L}} + Z_0 B = 2v^+$$

At $t \to \infty$ the exponential term is zero, and we have

$$B = \frac{2v^+}{Z_0}$$

At $t = 0$ the current is zero, so $A + B = 0$, $A = -B$. We thus have

$$i_T = \frac{2v^+}{Z_0}\left(1 - e^{-\frac{Z_0 t}{L}}\right)$$

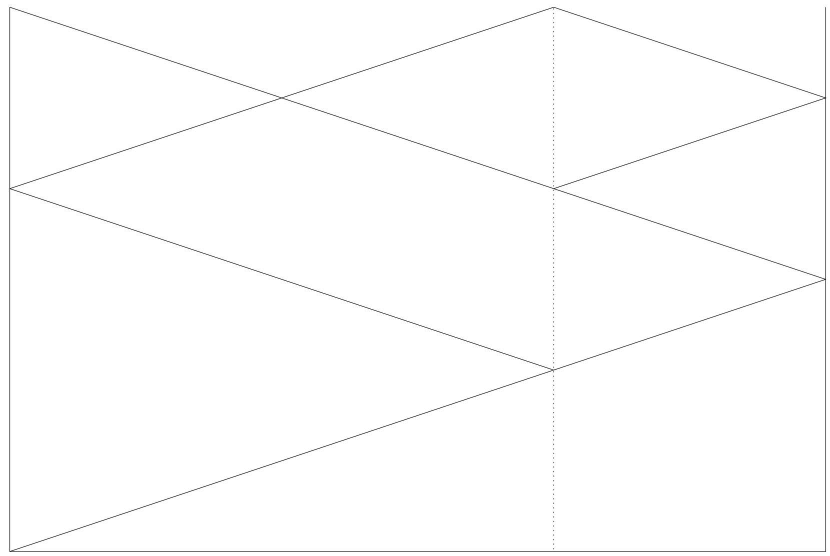
Now, $v^+ = \frac{Z_0}{Z_0+R_0}2V_0 = \frac{1}{2}2V_0 = V_0$, so

$$i_T = \frac{2V_0}{Z_0}\left(1 - e^{-\frac{Z_0 t}{L}}\right)$$

7.6. A transmission line of characteristics impedance $Z_0 = 50\,\Omega$ and length $d = 900\,\mathrm{m}$ is terminated by a load resistance of $200\,\Omega$. At a distance $z = 600\,\mathrm{m}$, a resistive discontinuity is introduced as shown in Figure P7.6, where $R_1 = R_2 = 50\,\Omega$, and $R_P = 100\,\Omega$. The line is then connected at $t = 0$ to a battery of $3\,\mathrm{V}$, which has an internal resistance of $100\,\Omega$. If the velocity of the wave propagation along the transmission line is $u = 3 \times 10^8\,\mathrm{m/s}$, use the reflection diagram to determine the following:

1. **Sending-end voltage versus time, up to $t = 6 \times 10{-}6\,\mathrm{s}$.**
2. **Voltage distribution along the transmission line at time $t = 3.5 \times 10^{-6}\,\mathrm{s}$.**

1. Here is a un-labeled reflection diagram



At every interface we have

$$v_T = v^+ + v^- \qquad i_T = \frac{v^+}{Z_0} - \frac{v^-}{Z_0}$$

We next find the ratio $v_T/i_T = Z_L$ divide the two equations, and re-arrange to find the reflection coefficient off a load Z_L as

$$\Gamma = \frac{v^-}{v^+} = \frac{Z_L - Z_0}{Z_L - Z_0}$$

Reflection coefficient off the generator is

$$\Gamma_G = \frac{Z_G - Z_0}{Z_G + Z_0} = \frac{100 - 50}{100 + 50} = \frac{1}{3}$$

Reflection coefficient off the load is

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{200 - 50}{200 + 50} = \frac{3}{5}$$

Reflection coefficient off the interface is the same whether approaching it from the right of the left because of symmetry, so we call it simply

$$\Gamma = \frac{Z_i - Z_0}{Z_i + Z_0}$$

where $Z_i = R_1 + [(R_2 + Z_0)\,||R_P] = 100\,\Omega$, so we get

$$\Gamma = \frac{100 - 50}{100 + 50} = \frac{1}{3}$$

Note that Z_i is the relationship between a total applied voltage and total resulting current.

The transmission coefficient through the interface is the same regardless of which side it is approached from. If we apply a total voltage v_T at the left side of the interface, we see that the right side output is 1/4 of that, so

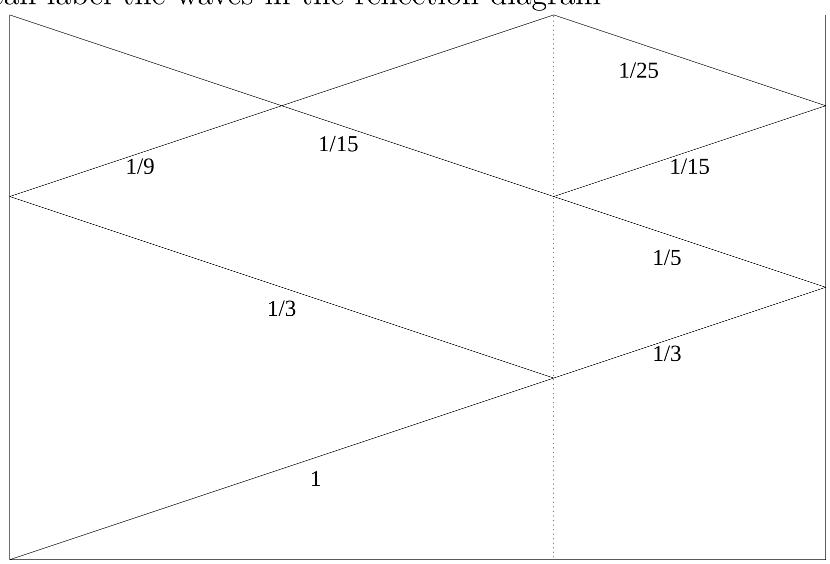
$$v_2^+ = \frac{v_1^+ + v_1^-}{4}$$

$$\tau = \frac{v_2^+}{v_1^+} = \frac{1 + \frac{v_1^-}{v_1^+}}{4} = \frac{1+\Gamma}{4} = \frac{\frac{4}{3}}{4} = \frac{1}{3}$$

This is also the transmission coefficient going in the other direction. The original incoming wave has amplitude

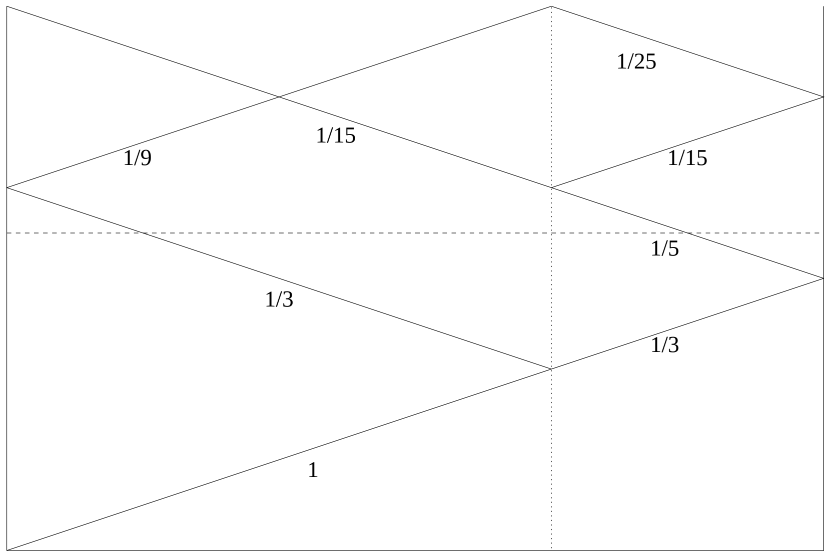
$$v_0 = 3 \times \frac{50}{100+50} = 1\,\text{V}$$

Now we can label the waves in the reflection diagram



For the first $4\,\mu$s the voltage is 1 V. For the next $2\,\mu$s the voltage is $1 + \frac{1}{3} + \frac{1}{9}\,\text{V} = 1.44\,\text{V}$. At $6\,\mu$s the voltage jumps to $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{15} + \frac{1}{45}\,\text{V} = 1.53\,\text{V}$, where the last term comes from the 1/3 reflection off the generator, but is not shown.

2. The voltage distribution along the line at that time can be found by summing all waves at each point up to that time



We see that the voltage distribution is as follows:

$$v(z) = \begin{cases} 1\,\mathrm{V} & 0 < z < 150\,\mathrm{m} \\ 1.33\,\mathrm{V} & 150 < z < 600\,\mathrm{m} \\ 0.33\,\mathrm{V} & 600 < z < 750\,\mathrm{m} \\ 0.53\,\mathrm{V} & 750 < z < 900\,\mathrm{m} \end{cases}$$