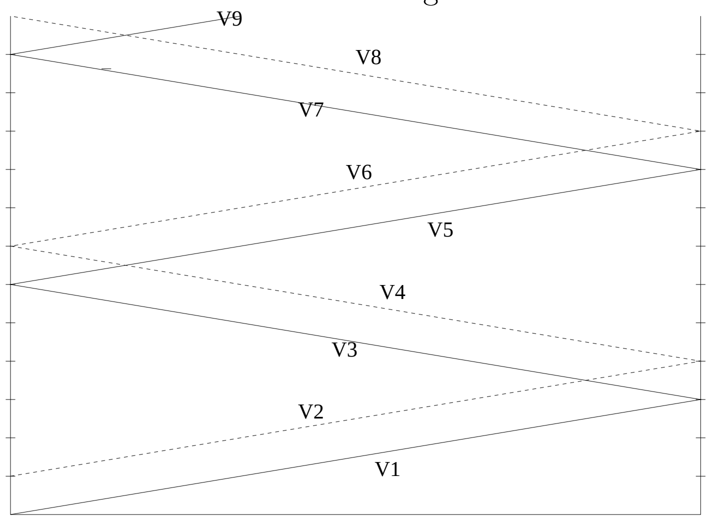
EE 333 Electricity and Magnetism, Fall 2009
Homework #13 solution

7.12. A pulse generator of output voltage $V_G = 1$, V and pulse duration of $1\,\mu$s is connected to a transmission line of characteristic impedance $Z_0 = 50\,\Omega$. and a velocity of propagation of $u = 3 \times 10^8$ m/s. The internal resistance of the pulse generator is $R_G = 75\,\Omega$, and the resistance terminating the transmission line is $R_L = 150\,\Omega$. The length of the transmission line is $d = 900$ m. Use the reflection diagram to determin the following:

1. **The voltage at the sending end as a function of time t up to $t = 12\,\mu$s.**
2. **The voltage at the load terminating the transmission line as a function of time up to time $t = 10\,\mu$s.**

1. The pulse consists of a positive transient followed by a negative transient. I draw the positive transient as a solid curve and the negative transient as a dashed curve.



The reflection coefficient at the load is

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{150 - 50}{150 + 50} = \frac{1}{2}$$

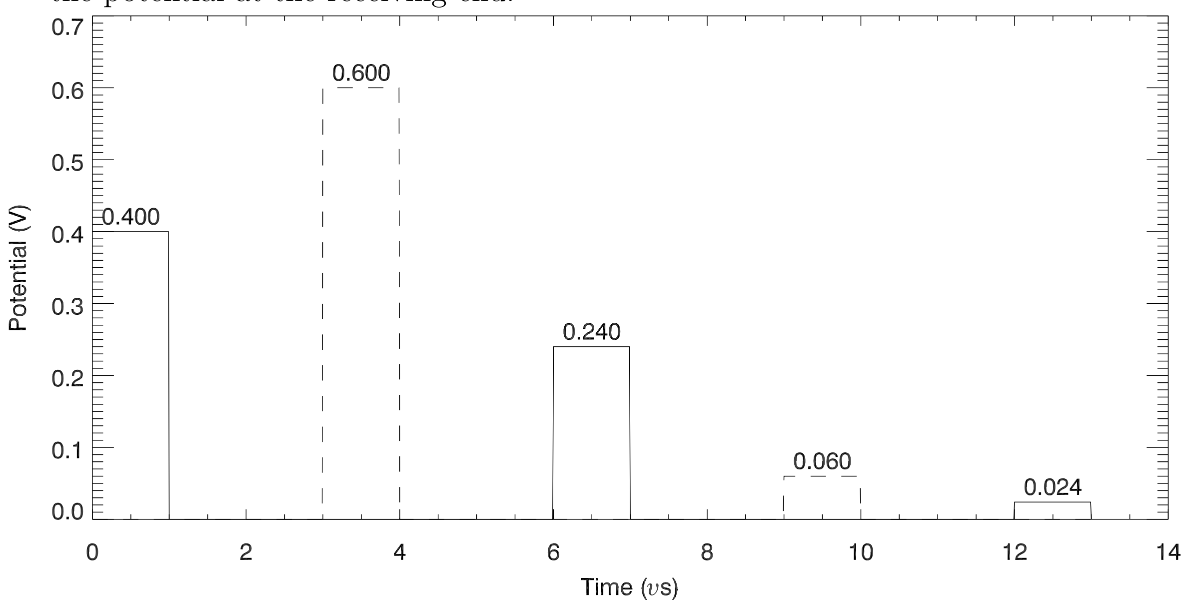
and at the generator it is

$$\Gamma_G = \frac{Z_G - Z_0}{Z_G + Z_0} = \frac{75 - 50}{75 + 50} = \frac{1}{5}$$

Then we have

$$
\begin{aligned}
V_1 =& V_G \frac{Z_0}{Z_G + Z_0} = 1 \times \frac{50}{50 + 75} = 0.4\,\text{V} \\
V_2 =& -V_1 = -0.4\,\text{V} \\
V_3 =& V_1 \Gamma_L = 0.4 \times 0.5 = 0.2\,\text{V} \\
V_4 =& -V_3 = -0.2\,\text{V} \\
V_5 =& V_3 \Gamma_G = 0.2 \times 0.2 = 0.04\,\text{V} \\
V_6 =& -V_5 = -0.04\,\text{V} \\
V_7 =& V_5 \Gamma_L = 0.04 \times 0.5 = 0.02\,\text{V} \\
V_8 =& -V_7 = -0.02\,\text{V} \\
V_9 =& V_7 \times \Gamma_G = 0.02 \times 0.2 = 0.004\,\text{V}
\end{aligned}
$$

2. Here are the waveforms. The solid curve is the potential at the sending end, the dashed the potential at the receiving end.



7.17. An antenna of input impedance $Z_L = 75 + j150\,\Omega$ at 2 MHz is connected to a transmitter through a 100-m section of coaxial cable, which has the following distributed constants: $r = 150\,\Omega/\text{km}$, $l = 1.4\,\text{mH/km}$, $c = 88\,\text{nF/km}$, and $g = 0.8\,\mu\text{S/km}$. If the output voltage of the transmitter is $V_G = 100e^{j0^\circ}$ V and its internal impedance is $75\,\Omega$ as shown in Figure P7-17, determine the following:

1. **The characteristic impedance Z_0 and the propagation constant $\gamma = \alpha + j\beta$ of the coaxial cable.**

2. **The input impedance Z_{in} of the cable terminated by the antenna.**

3. **The average power delivered to the load antenna.**

1. The characteristic impedance. We have

$$\begin{aligned} r + j\omega l =& 153 + j \times 2 \times \pi \times 2 \times 10^6 \times 1.4 \times 10^{-3} \\ =& 153 + j17593\,\Omega/\text{km} \end{aligned}$$

$$\begin{aligned} g + j\omega c =& 0.8 \times 10^{-6} + j \times 2 \times \pi \times 2 \times 10^6 \times 88 \times 10^{-9} \\ =& 0.8 \times 10^{-6} + j1.11\,\Omega^{-1}/\text{km} \end{aligned}$$

To a good approximation the characteristic impedance is then

$$Z_0 = \sqrt{\frac{l}{c}} = \sqrt{\frac{1.4 \times 10^{-3}}{88 \times 10^{-9}}} = 126\,\Omega$$

For low-loss lines we have

$$\begin{aligned} \alpha =& \frac{1}{2}\left(r\sqrt{\frac{c}{l}} + g\sqrt{\frac{l}{c}} \right) \\ =& \frac{1}{2}\left(150\sqrt{\frac{88 \times 10^{-9}}{1.4 \times 10^{-3}}} + 0.8 \times 10^{-6}\sqrt{\frac{1.4 \times 10^{-3}}{88 \times 10^{-9}}} \right) \\ =& 0.595\,\text{km}^{-1} \end{aligned}$$

$$\beta = \omega\sqrt{lc} = 2 \times \pi \times 2 \times 10^6 \times \sqrt{1.4 \times 10^{-3} \times 88 \times 10^{-9}} = 139\,\text{km}^{-1}$$

2. The reflection coefficient at the load is

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{75 + j150 - 126}{75 + j150 + 126} = -51 + j150201 + j150 = \frac{158e^{j109^\circ}}{250e^{j37^\circ}} = 0.632e^{j72^\circ}$$

We can then propagate this to a reflection coefficient at the transmitting end

$$\Gamma_G = \Gamma_L e^{-2\gamma d}$$

where $d = 100\,\text{m}$,

$$\gamma d = \alpha d + j\beta d = 0.595 \times 0.1 + j139 \times 0.1 = 0.0595 + j13.9 = 0.0595 + j796^\circ$$

$$\begin{aligned} \Gamma_G =& 0.632e^{j72^\circ}e^{-2\times 0.595}e^{-2j\times 796^\circ} \\ =& 0.561e^{j280^\circ} \end{aligned}$$

3. Next convert that into an impedance,

$$
\begin{aligned}
Z_{\text{in}} =& Z_0 \frac{1+\Gamma_G}{1-\Gamma_G} = 126 \frac{1+0.561e^{j280^\circ}}{1-0.561e^{j280^\circ}} = 126 \frac{1+0.974-0.552}{1-0.0974+0.552} \\
=& 126 \frac{1.0974-j0.552}{0.903+j0.552} = 126 \frac{1.228e^{-j27^\circ}}{1.058e^{j31^\circ}} = 146e^{-j58^\circ} \\
=& 77 - j124\,\Omega
\end{aligned}
$$

Now I will cheat a little and find total input power to the cable plus antenna, not just to the antenna (which would require finding voltage and current at the load end of the line).

$$
\begin{aligned}
V_{\text{in}} =& V_G \frac{Z_{\text{in}}}{Z_G + Z_{\text{in}}} = 100 \times \frac{146e^{-j58^\circ}}{75+77-j124} = 100 \frac{146e^{-j58}}{152-j124} = 100 \times \frac{146e^{-j58^\circ}}{196e^{-j32^\circ}} \\
=& 74.5e^{-j26^\circ}\ \text{V}
\end{aligned}
$$

The average input power is then

$$
P_{\text{in}} = \frac{1}{2}\text{Re}\left(VI^*\right) = \frac{1}{2}\text{Re}\left(\frac{V^2}{|Z_{\text{in}}|}\cos(\phi)\right) = \frac{1}{2}\frac{74.5^2}{146}\cos(58^\circ) = 10.1\ \text{W}
$$