Electromagnetic Fields and Waves

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PREFACE

Electromagnetic energy has highly diversified applications in communications, medicine, processing, and characterization of materials, biology, atmospheric sciences, radar systems, and in high-speed electronics and integrated circuits. Students in their junior or senior year of electrical engineering are expected to have either academically or in practice encountered applications involving electromagnetic fields, waves, and energy. For example, students should be familiar academically with electromagnetics in their introductory physics courses. Practical applications based on electromagnetics technology such as electric power lines, antennas, microwave ovens, and broadcast stations are encountered in our daily activities. Therefore, when students take electromagnetics courses they are expected to be excited and prepared to gain in-depth knowledge of this important subject. Instead, however, they quickly get bogged down with equations and mathematical relations involving vector quantities and soon lose sight of the interesting subject and exciting applications of electromagnetics.

It is true that the mathematical formulation of electromagnetics concepts is essential in quantifying the relationship between the electromagnetic fields and their sources. Integral and differential equations involving vector quantities are important in describing the characteristics and behavior of electromagnetic fields under a wide variety of propagation and interaction conditions. It is unfortunate, however, that the overall emphasis of the subject may be placed on these mathematical relations and their clever manipulation. Instead, the physical and exciting phenomena associated with electromagnetic radiation should be foremost, and mathematics should always be approached as a way to quantify and characterize electromagnetic fields, their radiation, propagation, and interactions. It is with this in mind that I have approached the development of this junior-level electrical engineering book on electromagnetic fields and waves.

There are several ways of organizing an introductory book on electromagnetics. One way is to start with the electrostatic and magnetostatic concepts, and continue to work toward the development of time-varying fields and dynamic electromagnetics. This has been the traditional procedure adopted in many textbooks. The other approach involves describing the mathematical relations between the time-varying electromagnetic fields and their sources by first introducing Maxwell's equations in integral forms. This allows a quick move toward the introduction of the propagation characteristics of plane waves. It is generally agreed that the second approach provides a faster pace toward the development of more exciting and dynamic aspects of electromagnetics, the subject matter that maintains high levels of enthusiasm for students and helps them carry on their otherwise difficult mathematical tasks.

I found the second method of organization to be helpful because students at the junior level usually have previous exposure to static fields. Also, the delay in discussing Maxwell's equations toward the end of the course does not help in consolidating and comprehending these important concepts and ideas. A few introductory textbooks adopt this approach. Although I used some of these books as texts when I initially taught the electromagnetics course series, I found it to be more constructive to include a concise description of the properties of the static electric and magnetic fields in terms of their charge and current sources before introducing Maxwell's equations. In addition, I have tried in this text to show how Maxwell's equations actually evolved from experimental observations made by Coulomb, Biot and Savart, Faraday and Ampere.

This brief introduction of the properties of electromagnetic fields and the experiments by pioneers in this field provides students with insight into the physical properties of these fields and help in developing a smoother transition from experimental observations to the mathematical relations that quantify them. In a sense, therefore, we may consider the adopted approach in this book to be a combination and a middle ground of the traditional approach of introducing the subject of electromagnetics in terms of static fields and the fast-paced approach of promptly introducing Maxwell's equations.

Additional features of this text are the inclusion of many examples in each chapter to help emphasize key concepts, detailed description of the subject of "reflection and refraction of plane waves of oblique incidence on a dielectric interface," including some of its applications in optics, and a detailed introduction to antennas including physical mechanisms of radiation and practical design of antenna arrays. The treatment of the subject of transmission lines was comprehensive and included a detailed treatment of transients and sinusoidal steady-state analysis of propagation on two conductor lines. Another important feature of this text is the introductory section on "numerical techniques" included in chapter 4. At this time and age, many solutions are handled by computers and, with the availability of this technology, solutions to more realistic and exciting engineering problems may be included in homework assignments and even simulated and demonstrated in classrooms. It is essential, however, that students be familiar with the commonly used computational procedures such as the finite difference method and the method of moments, learn of the various approximations involved, and be aware of the limitations of such methods. Recently, some focused efforts* have

* NSF/IEEE Center on Computer Applications in Electromagnetic Education (CAEME), University of Utah, Salt Lake City, UT 84112.

attempted to stimulate, accelerate, and encourage the use of computers and software tools to help electromagnetic education. Many educational software packages are now available to educators, and it is imperative that students be aware of the capabilities, accuracies, and limitations of some of these software tools—particularly those that use computational techniques and numerical methods. It is with this in mind that we prepared the introductory material on computational methods in chapter 4. Furthermore, educators and students are encouraged to use available software from CAEME* to help comprehend concepts, visualize the dynamic-field phenomena, and solve interesting practical applications.

I would like to conclude by expressing my sincere thanks and appreciation to my students who, during the years, provided me with valuable feedback on the manuscript. Comments and suggestions by Professor Robert S. Elliott of University of California, Los Angeles, were deeply appreciated. I would also like to express my sincere appreciation to Ruth Eichers and Holly Cox for their expert efforts in typing and preparing the manuscript. My gratitude, sincere thanks, deep appreciation, and love are also expressed to my family for patience, sacrifice, and understanding during the completion of this endeavor.

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CHAPTER 1

VECTOR ANALYSIS AND MAXWELL'S EQUATIONS IN INTEGRAL FORM

1.1 INTRODUCTION

In this chapter we will first review some simple rules of vector algebra. These basic vector operations are first defined independent of any coordinate system and then specifically applied to the Cartesian, cylindrical, and spherical coordinate systems. Transformation of vector representation from one coordinate system to another will also be described. Scalar and vector fields will then be defined, with emphasis on understanding the concepts of electric and magnetic fields because they constitute the basic elements of electromagnetics. Vector integration will be introduced to pave the way for the introduction of Maxwell's equations in integral form. Maxwell's equations are simply the mathematical relations that govern the relationships between the electric and magnetic fields, and their associated charge and current distribution sources. These relations include the following:

1. Gauss's law for the electric field.
2. Gauss's law for the magnetic field.
3. Faraday's law.
4. Ampere's circuital law.

A brief description of the experimental evidence that led to Maxwell's hypothesis will also be given.

1.2 VECTOR ALGEBRA

Familiarity with some of the mathematical rules of the vector calculus certainly helps in simplifying the development of the electromagnetic fields theory. This is simply because the electric and magnetic fields, which are the bases of our study, are vector quantities, the matter that makes it useful for us to start with reviewing our vector algebra. Let us first distinguish between scalar and vector quantities.

Scalar: Is a physical quantity completely specified by a single number describing the magnitude of the quantity (e.g., temperature, size of a class, mass, humidity, etc.).

Vector: Is a physical quantity that can only be specified if both magnitude and direction of the quantity are given. This class of physical quantities *cannot* be described by *one number* only (e.g., force field, velocity of a car or a tornado, etc.).

Graphically, a vector is represented as shown in Figure 1.1 by a straight line with an arrowhead pointing in the direction of the vector and of length proportional to the magnitude of the vector.

Unit Vector: A unit vector in a given direction is a vector along the described direction with magnitude equal to unity.

In Figure 1.2, $\mathbf{A}$ is a vector along the x axis, and $\mathbf{a}_x$ is a unit vector along the x axis.

$$\mathbf{a}_x = \frac{\mathbf{A}}{|\mathbf{A}|}$$

Hence, any vector can be represented as a product of a unit vector in the direction of the vector with the magnitude of the vector

$$\mathbf{A} = |\mathbf{A}|\,\mathbf{a}_x$$

Figure 1.1 Vector representation by an arrow. The length of the arrow is proportional to the magnitude of the vector, and the direction of the vector is indicated by the direction of the arrow.

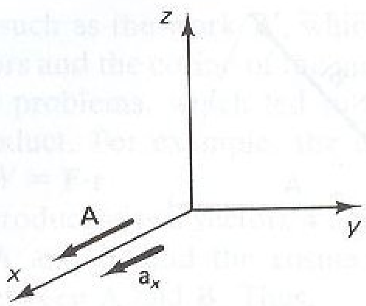

Figure 1.2 A unit vector $\mathbf{a}_x$ along the direction of the vector **A**.

1.2.1 Vector Addition and Subtraction

Four possible types of vector algebraic operations exist. This includes vector additions, subtractions, scalar, and vector products. In the following two sections, we will discuss these operations in more detail. Let us start with the process of adding and subtracting vector quantities.

The displacement of a point for a certain distance along a straight line is a good illustration of a physical vector quantity. For example, the displacement of a point from location 1 to location 2 in Figure 1.1 represents a vector quantity where its magnitude equals the distance between the end points 1 and 2, and the vector direction is along the straight line connecting 1 to 2. The addition of two vectors, therefore, can be described as the net displacement that results from two consecutive displacements. In Figure 1.3, vector **A** represents the vector displacement between 1 and 2, whereas the vector **B** represents the vector displacement between 2 and 3. The total displacement between 1 and 3 is described by the vector **C**, which is the sum of the individual displacements **A** and **B**. Hence,

$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

Based on similar reasoning, it is fairly simple to show that

$$(\mathbf{A} + \mathbf{B}) + \mathbf{D} = \mathbf{A} + (\mathbf{B} + \mathbf{D})$$

Because the negative of a vector is defined as a vector with the same magnitude but opposite direction, *vector subtraction* can be easily defined in terms of vector addition. In other words, the subtraction of two vectors can be thought of as the summation of one vector and the negative of the other,

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$$

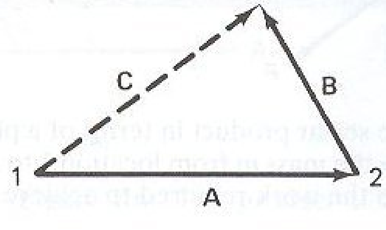

Figure 1.3 The vector addition of two displacements.

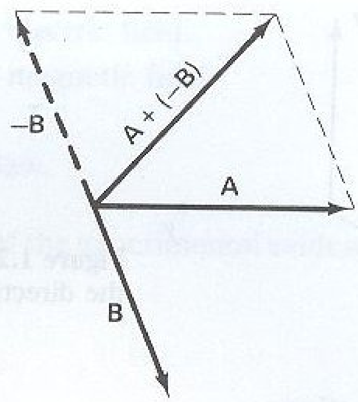

Figure 1.4 Vector subtraction performed as the addition of one vector to the negative of the other.

Figure 1.4 illustrates the process of vector subtraction where it is shown that a vector $-\mathbf{B}$ was first obtained and then added to the vector $\mathbf{A}$ to provide the resultant vector $\mathbf{A} + (-\mathbf{B})$.

1.2.2 Vector Multiplication

The process of vector multiplication is more involved than the simple multiplication of scalar quantities. The directions of the vectors are involved in the multiplication process, which further complicates the procedure. Two kinds of multiplications are commonly encountered in physical problems and hence are given special shorthand notations. These are the scalar (dot) product and the vector (cross) product. In the following sections, these two vector product procedures will be explained in more detail.

Scalar (dot) product of two vectors. The name "scalar product" emerged from the fact that the result of this multiplication process is a scalar quantity. To appreciate the physical reasoning behind the scalar product of two vectors, let us assume an object of mass m placed on a rough surface s. To move this object from location 1 to location 2, a vector force $\mathbf{F}$, which makes an angle α with respect to the displacement vector $\mathbf{r}$, is applied as shown in Figure 1.5. It is required to calculate the work done in moving m from location 1 to 2. This work W is actually equal to the component of the force along the direction of motion multiplied by the distance between 1 and 2, hence

$$W = |\mathbf{F}| \cos\alpha \, |\mathbf{r}|$$

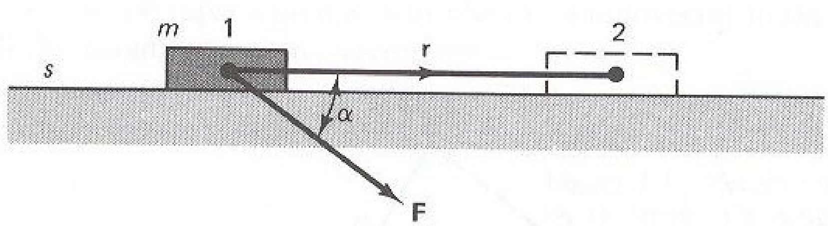

Figure 1.5 Explanation of the scalar product in terms of a physical problem. Force $\mathbf{F}$ is applied to move the mass m from location 1 to 2. The scalar product of $\mathbf{F}$ and $\mathbf{r}$ is related to the work required to achieve this motion.

Scalar quantities, such as the work W, which are calculated by multiplying the magnitudes of two vectors and the cosine of the angle between them, are encountered in many other physical problems, which led to identifying them by the shorthand notation of the dot product. For example, the desired work in Figure 1.5 may be expressed in the form $W = \mathbf{F}\cdot\mathbf{r}$.

The scalar or dot product of two vectors **A** and **B** is therefore equal to the product of the magnitudes of **A** and **B**, and the cosine of the angle between them. It is represented by a dot between **A** and **B**. Thus,

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\alpha = AB \cos\alpha$$

where α is the angle between **A** and **B**.

The dot product operation can also be interpreted as the multiplication of the magnitude of one vector by the scalar obtained by projecting the second vector onto the first vector as shown in Figure 1.6.

The dot product can therefore be expressed as $\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\alpha = |\mathbf{A}|$ multiplied by the projection of **B** along **A** (i.e., $|\mathbf{B}| \cos\alpha$ as shown in Figure 1.6b) $= |\mathbf{B}|$ multiplied by the projection of **A** along **B** (i.e., $|\mathbf{A}| \cos\alpha$ as shown in Figure 1.6a). Based on this interpretation, it may be emphasized that the dot product of two perpendicular vectors is zero. This can be seen by simply noting that the projection of one vector along the other that is perpendicular to it is zero. Such an observation is usually more useful than going through the mathematical substitution and recognizing that the angle α between the two perpendicular vectors is $\pi/2$ and that $\cos \pi/2 = 0$.

The distributive property for the dot product of the sum of two vectors with a third vector is:

$$\mathbf{A}\cdot(\mathbf{B} + \mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$$

Figure 1.7 illustrates that the projection of $\mathbf{B} + \mathbf{C}$ onto **A** is equal to the sum of the individual projections of **B** and **C** onto **A**.

Vector (cross) product of two vectors. The vector or cross product of two vectors **A** and **B** is a *vector*, perpendicular to **A** and **B** or equivalently perpendicular to the plane containing **A** and **B**. The direction of the vector product is obtained by the right-hand rule rotating the first vector **A** to coincide with the second vector **B** in the

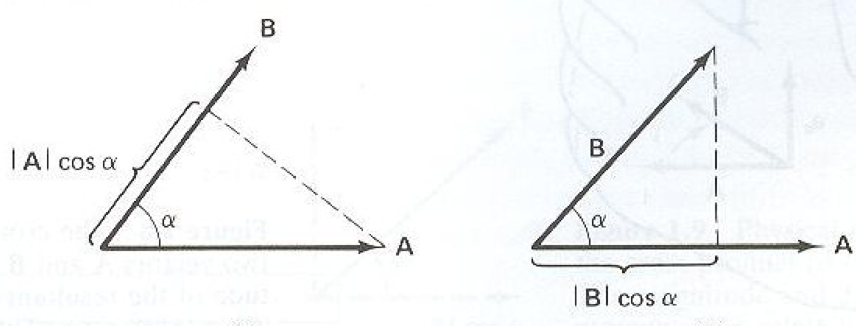

Figure 1.6 Dot product of two vectors.

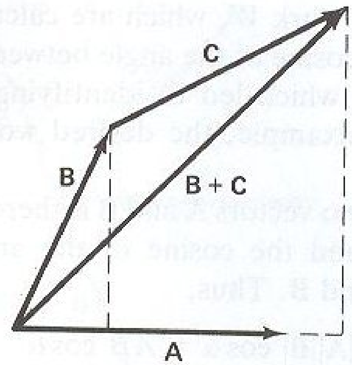

Figure 1.7 The distributive property for the dot product.

shortest way (through the angle α of Figure 1.8a). The magnitude of the cross product of two vectors is obtained by multiplying the magnitudes of the two individual vectors and sine of the angle between them. Figure 1.8 shows the magnitude and direction of vector **C**, which resulted from the cross product of **A** and **B**

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = AB \sin\alpha\, \mathbf{a}_c$$

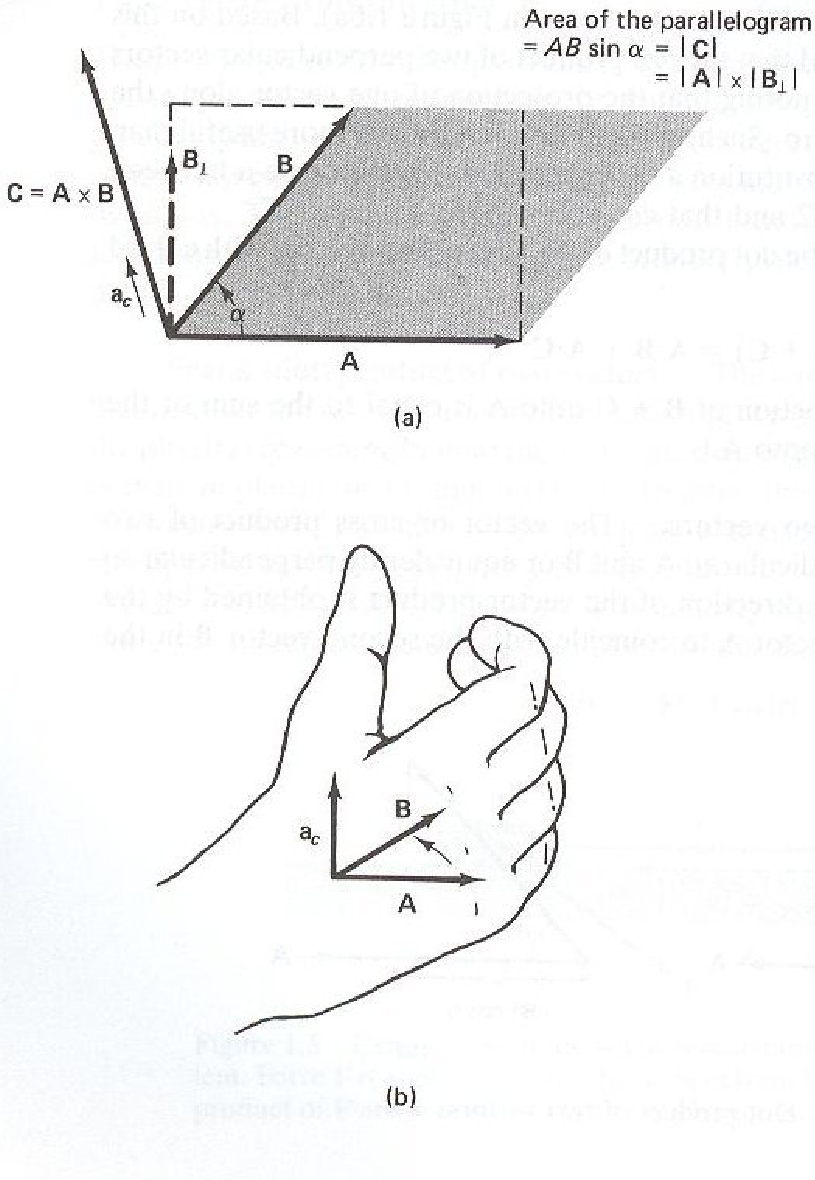

Figure 1.8 The cross product of two vectors **A** and **B**. The magnitude of the resultant vector **C** is $|\mathbf{C}| = |\mathbf{A}||\mathbf{B}| \sin\alpha$. The direction of **C** is obtained according to the right-hand rule shown in b.

where $\mathbf{a}_c$ is a unit vector perpendicular to **A** and **B** and in the direction indicated by the right-hand rule shown in Figure 1.8b.

To illustrate the importance of the cross product in physical problems, let us consider the lever ℓ that is free to rotate around a pivot O. A force **F** is applied to the lever at point a as shown in Figure 1.9. It is required to calculate the moment **M** of the force **F** around the pivot O. From Figure 1.9, it is clear that the moment **M** is actually related to the component of **F** perpendicular to **r**—that is, $|\mathbf{F}| \sin\alpha$. The other component of **F** in the direction of **r** does not contribute to the rotation of the lever around O. The magnitude of the moment $|\mathbf{M}|$ is therefore given by

$$|\mathbf{M}| = |\mathbf{F}| \sin\alpha \, |\mathbf{r}|$$

Figure 1.9 shows that in certain physical problems parameters of interest, such as the moment in our case, are obtained by multiplying the magnitudes of two vectors by the sine of the angle between them. The magnitude of the moment, however, does not provide a *complete description* of the amount and direction of rotation of the lever. An indication of the direction of the moment is still required. To obtain the direction of the moment, it may be seen from Figure 1.9 that for the indicated direction of the force **F** the rotation of the lever will be in the counterclockwise direction. Therefore, if we imagine the presence of a screw at O, it can be seen that such a screw will proceed in the direction out of the plane of the paper as a result of the rotation. The direction to which a screw proceeds as a result of the rotation is taken to be the direction of the moment **M**. From Figure 1.9, it may be seen that such a direction is the same as that obtained according to the right-hand rule when applied to the vectors **r** and **F** in the sequence from **r** to **F**. Hence, a complete description of the moment **M** (i.e., magnitude and direction) is given by

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

in which case the magnitude of **M** is obtained by multiplying the magnitudes of **r** and **F** by the sine of the angle α, and the direction of **M** is indicated by the right-hand rule from **r** to **F** as explained earlier.

Therefore, the shorthand notation of the cross product of two vectors **A** and **B** is simply a vector with its magnitude equal to $|\mathbf{A}||\mathbf{B}| \sin\alpha$, where α is the angle between **A** and **B**, and the direction of the resultant vector is obtained according to the right-hand rule shown in Figure 1.8b.

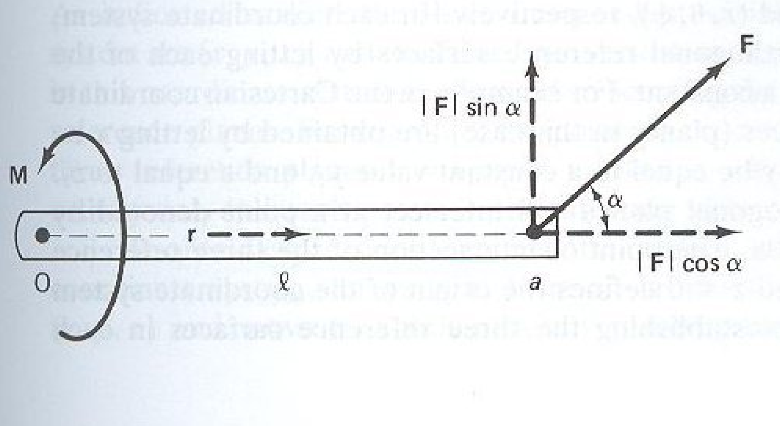

Figure 1.9 Physical illustration of the cross product of two vectors. The magnitude and direction of the moment **M** is related to the cross product of the force vector **F** and the distance vector **r**, $\mathbf{M} = \mathbf{r} \times \mathbf{F}$.

Another physical interpretation of the cross product can be made in terms of the vector projections. For example, the vector **C** in Figure 1.8a is given by

$$\mathbf{C} = |\mathbf{A}||\mathbf{B}| \sin\alpha\, \mathbf{a}_c$$
$$= \mathbf{A} \times \mathbf{B}_\perp = |\mathbf{A}||\mathbf{B}_\perp|\, \mathbf{a}_c$$

where $\mathbf{B}_\perp$ is the vector component of **B** perpendicular to **A**. This observation simply indicates that the cross product of two vectors involves the multiplication of one vector (e.g., **A**) by the component of the other perpendicular to it. Based on this observation, it is useful to note that the cross product of two vectors that are in the same direction (i.e., parallel vectors) is zero. This may be seen by either noting that the angle α between two parallel vectors is zero and hence $\sin\alpha = 0$, or by recognizing that for parallel vectors the component of one vector perpendicular to the other is zero. The usefulness of such observations will be clarified in later discussions.

From the right-hand rule of Figure 1.8b, it is rather straightforward to see that

$$\mathbf{B} \times \mathbf{A} = -\mathbf{C} = -\mathbf{A} \times \mathbf{B}$$

which means that the ordering of the vectors in the cross product is an important consideration because the cross product does not obey a commutative law.

1.3 COORDINATE SYSTEMS

The vectors and the vector relations given in the previous sections are not defined with respect to any particular coordinate system. Hence, all the previously indicated definitions of the dot product, cross product, and so forth are presented in graphical and general terms.

Having a certain reference system (known as the coordinate system), however, is important to describe *uniquely* the position of a point in space, and the magnitude and direction of a vector. Although several coordinate systems are available, we will restrict our discussion to the three simplest ones—namely, the so-called Cartesian, cylindrical, and spherical coordinate systems. Expressions for transforming a vector representation from one coordinate system to another will be derived and the previously defined vector algebraic relations will be given in these three coordinate systems.

To start with, each of the three coordinate systems is specified in terms of three independent variables. In the Cartesian coordinate system these independent variables are (x, y, z), whereas for the cylindrical and spherical coordinate systems these independent variables are (ρ, ϕ, z) and (r, θ, ϕ), respectively. In each coordinate system, we also set up three mutually orthogonal reference surfaces by letting each of the independent variables be equal to a constant. For example, in the Cartesian coordinate system, the three reference surfaces (planes in this case) are obtained by letting x be equal to a constant value, say x_1, y be equal to a constant value y_1, and z equal to z_1. As a result, these mutually orthogonal planes will intersect at a point denoted by (x_1, y_1, z_1) as shown in Figure 1.10a. The point of intersection of the three reference planes for which $x = 0$, $y = 0$, and $z = 0$ defines the origin of the coordinate system as shown in Figure 1.10b. After establishing the three reference surfaces in each

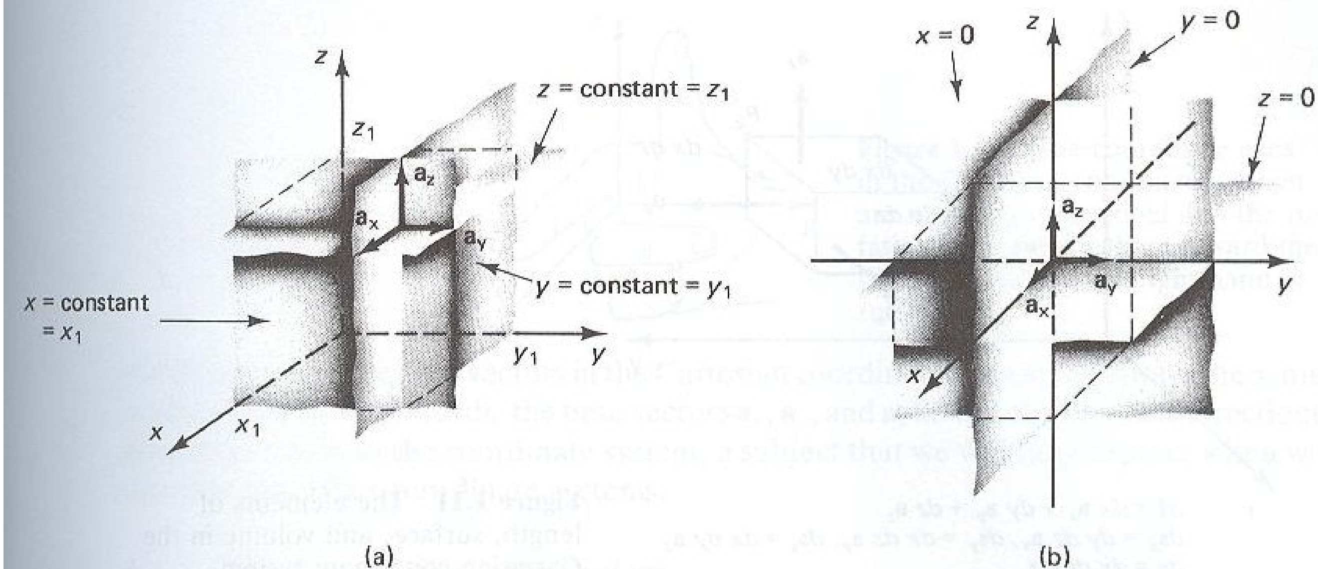

Figure 1.10 The Cartesian coordinate system. (a) The point (x_1, y_1, z_1) is generated at the intersection of $x = x_1$ plane with the $y = y_1$ and $z = z_1$ planes. (b) The origin is the point of intersection of $x = 0$, $y = 0$, and $z = 0$ planes. The base vectors $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ are mutually orthogonal, and each is perpendicular to a reference plane.

coordinate system, we define three mutually orthogonal unit vectors, called the *base vectors*. The directions of these base vectors are chosen such that each base vector is perpendicular to a reference surface and oriented in the direction of increasing the independent variable. For example, the base vector $\mathbf{a}_x$ shown in Figure 1.10b is oriented perpendicular to the $x =$ constant plane and is in the direction of increasing x. Similarly the base vectors $\mathbf{a}_y$ and $\mathbf{a}_z$ are oriented perpendicular to the $y =$ constant and $z =$ constant planes, respectively. Any vector is represented in a coordinate system in terms of its components along the base vectors of that system. For example, in the Cartesian coordinate system, a vector **A** should be represented in terms of its components A_x, A_y, A_z along the unit (base) vectors $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$. These, as well as other characteristics of the three coordinate systems, will be described in the following sections.

1.3.1 Cartesian Coordinate System

As indicated earlier, the three independent variables in the Cartesian coordinate system are (x, y, z), and the three base vectors are $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$. The location of a point in this coordinate system is obtained by locating the point of intersection of the three reference planes. For example, the point (x_1, y_1, z_1) is the point of intersection of the three reference planes $x = x_1$, $y = y_1$, and $z = z_1$. The base vectors are mutually orthogonal, and each points in the direction of increase of an independent variable.

To obtain expressions for elements of length, surface, and volume in the Cartesian coordinate system, let us start from an arbitrarily located point P_1 of coordinates (x, y, z) and move to another closely placed point P_2 of coordinates $(x + dx, y +$

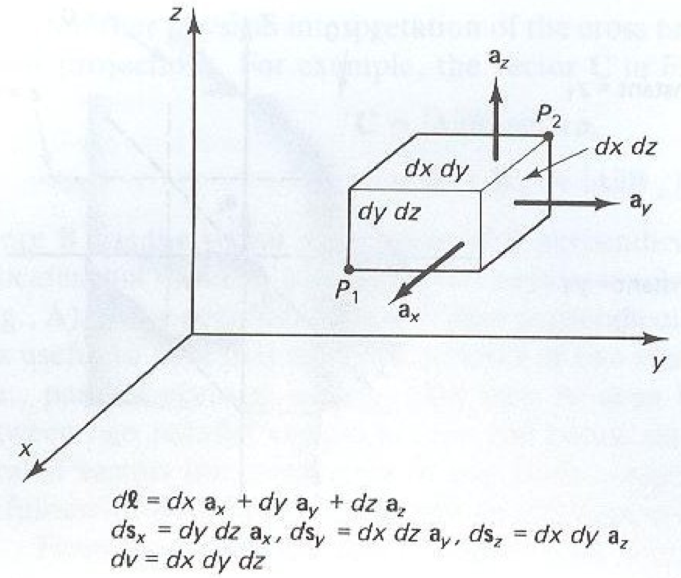

Figure 1.11 The elements of length, surface, and volume in the Cartesian coordinate system.

$dy, z + dz$) as shown in Figure 1.11. Thus, in moving from P_1 to P_2, we basically changed the values of the independent variables from x to $x + dx$, y to $y + dy$, and from z to $z + dz$. The element of volume, dv, generated from these incremental changes in the independent variables is given, as shown in Figure 1.11, by

$$dv = dx\,dy\,dz$$

The *vector* element of length, $d\ell$, between P_1 and P_2, conversely, should be expressed, like any other vector, in terms of its components along the three mutually orthogonal base vectors. From Figure 1.11, it can be shown that $d\ell$ has a component, dx, along the $\mathbf{a}_x$ base vector, dy along the $\mathbf{a}_y$, and dz along the $\mathbf{a}_z$ unit vector. Therefore, $d\ell$ may be expressed as

$$d\ell = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z$$

Regarding the elements of area, it is important to emphasize that each element of area should be accompanied by a unit vector specifying its orientation in the coordinate system. For example, it is not sufficient to indicate an element of area ds_x equal to $dy\,dz$ because it leaves the orientation or the direction of this element of area unspecified. As a result, we can specify three elements of areas in the Cartesian coordinate system as

$$ds_x = dy\,dz\,\mathbf{a}_x$$

$$ds_y = dx\,dz\,\mathbf{a}_y$$

$$ds_z = dx\,dy\,\mathbf{a}_z$$

where each element of area is specified by a unit vector perpendicular to it. Actually the subscripts are not necessary to include in this case but are here just to emphasize that ds_x (subscript x) is an element of area in the $\mathbf{a}_x$ direction and so on.

It should be noted that the three coordinate axes x, y, and z are oriented with respect to each other according to the right-hand rule as shown in Figure 1.12 and that

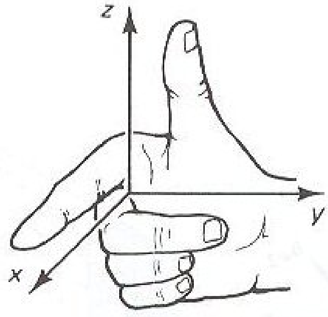

Figure 1.12 The coordinate axes in the Cartesian coordinate system are mutually orthogonal and the rotation from two of them toward the third axis follows the right-hand rule.

the directions of the base vectors in the Cartesian coordinate system are always the same at all points. In other words, the base vectors $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ do not change their directions at various points in the coordinate system, a subject that we will fully explore when we describe the other coordinate systems.

1.3.2 Cylindrical Coordinate System

In this coordinate system the three independent variables are ρ, ϕ, and z. The three reference surfaces are a cylindrical surface generated by letting ρ = constant = ρ_1, and two plane surfaces obtained from ϕ = constant = ϕ_1 and z = constant = z_1. These three reference planes intersect at the coordinate point (ρ_1, ϕ_1, z_1). The origin of the coordinate system is the point of the intersection of the three reference planes for which the values of the independent variables are all equal to zero. Figure 1.13 shows the reference surfaces in the cylindrical coordinate system.

The three base vectors $\mathbf{a}_\rho$, $\mathbf{a}_\phi$, and $\mathbf{a}_z$ are also shown in Figure 1.13 where it is clear that these vectors are oriented perpendicular to the reference surfaces—that is, $\mathbf{a}_\rho$ is perpendicular to the ρ = constant cylindrical surface, $\mathbf{a}_\phi$ is perpendicular to the plane

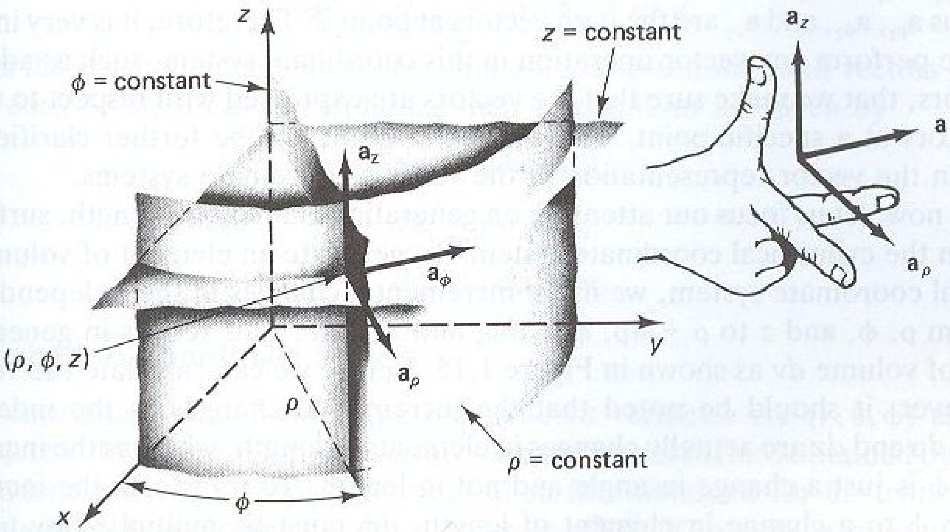

Figure 1.13 The cylindrical coordinate system. The three reference planes intersect at the point (ρ, ϕ, z), and the three base vectors are $\mathbf{a}_\rho$ normal to the cylindrical surface, ρ = constant, $\mathbf{a}_\phi$ normal to the ϕ = constant plane, and $\mathbf{a}_z$ is normal to the z = constant plane.

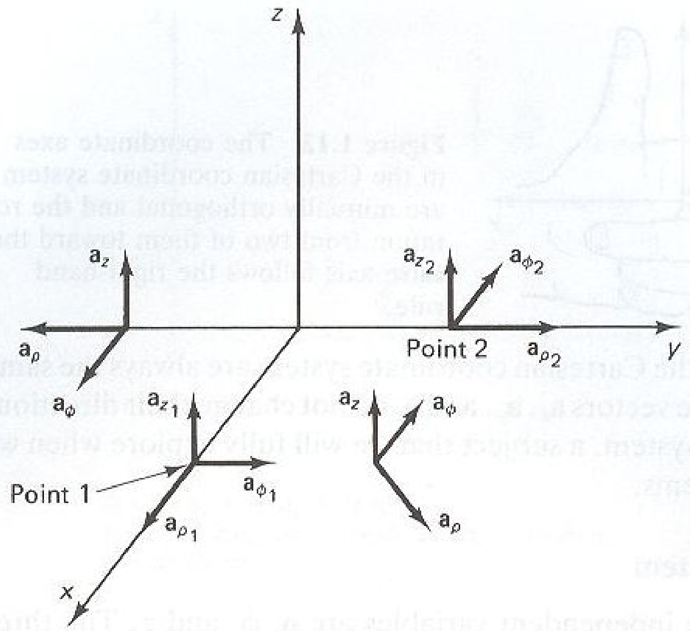

Figure 1.14 The base vectors in the cylindrical coordinate system change directions at the various points. The base vectors at point 1 are $\mathbf{a}_{\rho 1}$, $\mathbf{a}_{\phi 1}$, and $\mathbf{a}_{z1}$, whereas $\mathbf{a}_{\rho 2}$, $\mathbf{a}_{\phi 2}$, and $\mathbf{a}_{z2}$ are the base vectors at point 2.

ϕ = constant, and $\mathbf{a}_z$ is perpendicular to the z = constant plane—and that all the base vectors point in the direction of the increase in the independent variables. Figure 1.14 shows the directions of the base vectors at various points—that is, various values of ρ, ϕ, and z, in the cylindrical coordinate system. From Figure 1.14, it may be seen that, unlike the base vectors in the Cartesian coordinate system, the base vectors in the cylindrical coordinate system do not maintain their same directions at the various points. For example, we note that the base vectors at a point 1 along the x axis, that is, $\phi = 0$, are related to those at a point 2 along the y axis, that is, $\phi = \pi/2$, by $\mathbf{a}_{\rho_1} = -\mathbf{a}_{\phi_2}$, $\mathbf{a}_{\phi_1} = \mathbf{a}_{\rho_2}$, and $\mathbf{a}_{z_1} = \mathbf{a}_{z_2}$ where $\mathbf{a}_{\rho_1}$, $\mathbf{a}_{\phi_1}$, and $\mathbf{a}_{z_1}$ are the base vectors at point 1, whereas $\mathbf{a}_{\rho_2}$, $\mathbf{a}_{\phi_2}$, and $\mathbf{a}_{z_2}$ are the base vectors at point 2. Therefore, it is very important before we perform any vector operation in this coordinate system, such as addition of two vectors, that we make sure that the vectors are expressed with respect to the *same* base vectors at a specific point. This particular point will be further clarified in the section on the vector representation in the various coordinate systems.

For now, let us focus our attention on generating elements of length, surface, and volume in the cylindrical coordinate system. To generate an element of volume in the cylindrical coordinate system, we make incremental changes in the independent variables from ρ, ϕ, and z to $\rho + d\rho$, $\phi + d\phi$, and $z + dz$. This results in generating an element of volume dv as shown in Figure 1.15. Before we can calculate the volume of dv, however, it should be noted that the incremental changes in the independent variables $d\rho$ and dz are actually changes in elements of length, whereas the incremental change $d\phi$ is just a change in angle and not in length. To transform the incremental change in ϕ to a change in element of length, $d\phi$ must be multiplied by ρ and the corresponding change in the linear dimension will be $d\ell_\phi = \rho d\phi\, \mathbf{a}_\phi$ as shown in Figure 1.15. In other words, we multiplied $d\phi$ by ρ, which is called the metric coefficient to transform the change in the angle $d\phi$ to change in the linear dimension $d\ell_\phi$. With this

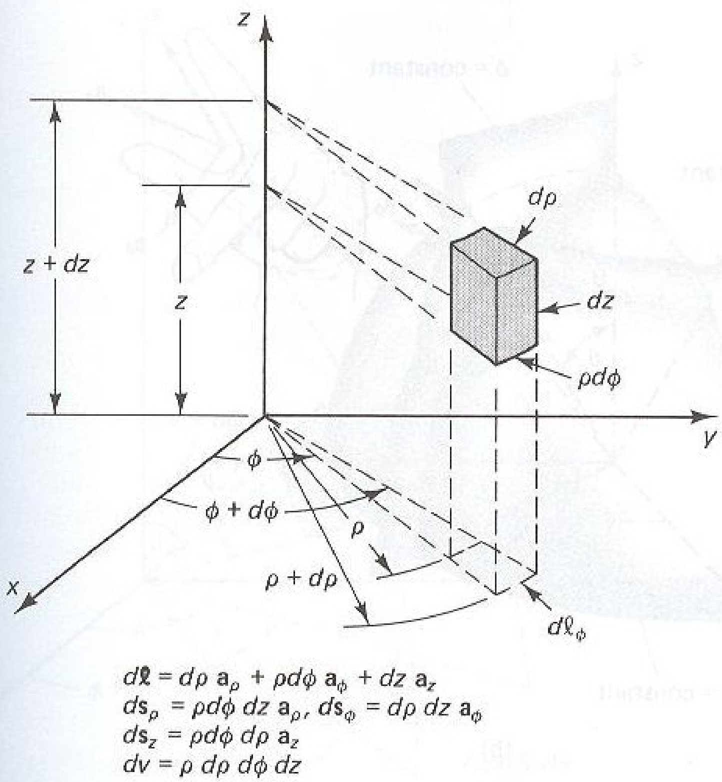

Figure 1.15 The elements of length, volume, and surface in the cylindrical coordinate system. $d\phi$ is not a length and should be multiplied by its metric coefficient ρ to have the dimension of length. $\therefore\ d\ell_\phi = \rho d\phi\, \mathbf{a}_\phi$.

in mind, it is easy to show that the element of volume dv, which resulted from incremental changes in the independent variables, is given by

$$dv = d\rho(\rho d\phi)dz = \rho\, d\rho\, d\phi\, dz$$

The resultant element of length $d\ell$ from P_1 to P_2 is given by

$$d\ell = d\rho\, \mathbf{a}_\rho + \rho d\phi\, \mathbf{a}_\phi + dz\, \mathbf{a}_z$$

whereas the resultant elements of area that are associated with unit vectors perpendicular to each of the areas to emphasize their orientations are given by

$$d\mathbf{s}_\rho = \rho d\phi\, dz\, \mathbf{a}_\rho$$

$$d\mathbf{s}_\phi = d\rho\, dz\, \mathbf{a}_\phi$$

$$d\mathbf{s}_z = \rho d\rho\, d\phi\, \mathbf{a}_z$$

1.3.3 Spherical Coordinate System

In this coordinate system the three independent variables are (r, θ, ϕ) as shown in Figure 1.16a. The three reference surfaces are: spherical surface obtained by letting the independent variable r = constant, conical surface obtained for θ = constant value, and a plane surface obtained for ϕ = constant value. These three reference surfaces intersect at the coordinate point $P\,(r, \theta, \phi)$ as shown in Figure 1.16b. The three base vectors $\mathbf{a}_r$, $\mathbf{a}_\theta$, and $\mathbf{a}_\phi$ are perpendicular to the spherical, conical, and the plane reference surfaces, respectively. These three base vectors are clearly mutually orthogonal, and

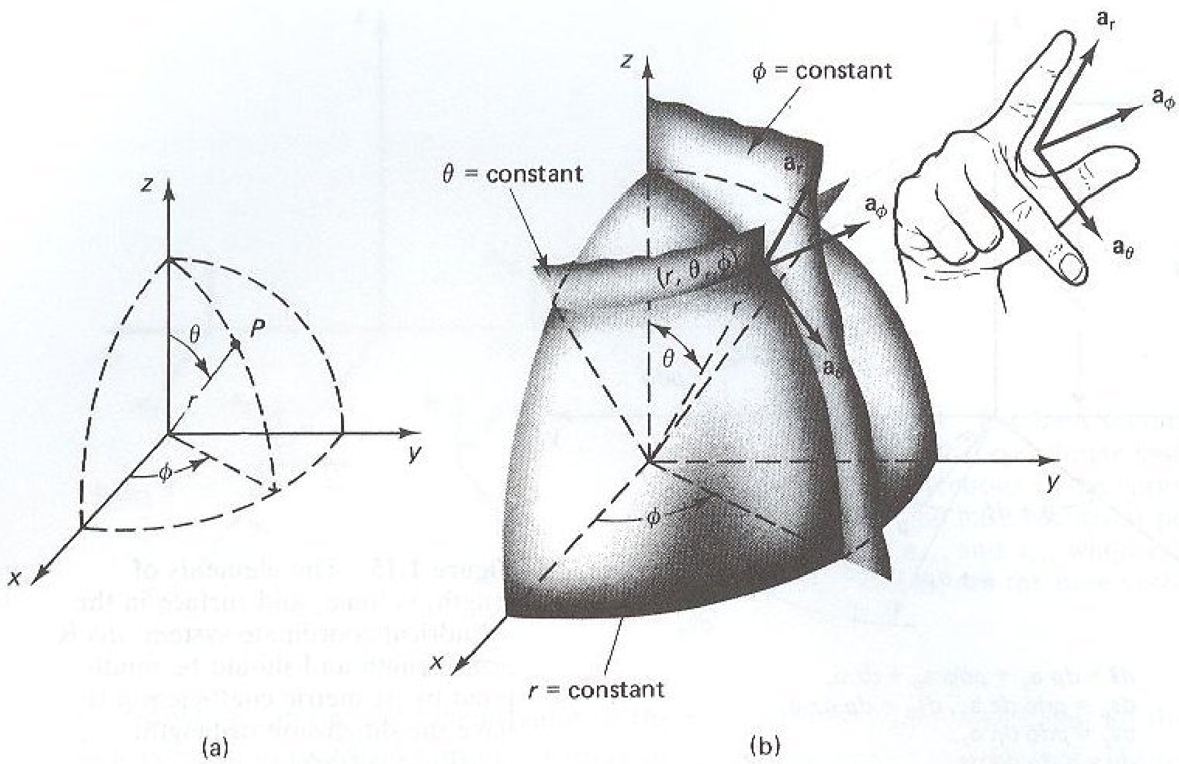

Figure 1.16 The spherical coordinate system. (a) The three independent variables (r, θ, ϕ) at point P. (b) The three reference surfaces are the spherical surface r = constant, the conical surface θ = constant, and the plane ϕ = constant. The three base vectors $\mathbf{a}_r$, $\mathbf{a}_\theta$, and $\mathbf{a}_\phi$ are mutually orthogonal and follow the right-hand rule.

they point in the directions of the increase of the independent variables. The orientation of the base vectors is in accordance to the right-hand rule as also shown in Figure 1.16b. The differential elements of volume, surface, and length are routinely generated by incrementally changing the independent variables from r, θ, and ϕ to $r + dr$, $\theta + d\theta$, and $\phi + d\phi$ as shown in Figure 1.17. Expressions for the differential elements are obtained by noting that the incremental changes in the independent variables $d\theta$ and $d\phi$ are not actual changes in elements of length, but instead are just changes in angles. To transform the change $d\theta$ into a change in a differential element of length, $d\theta$ must be multiplied by the metric coefficient which is, in this case, r. In other words, the incremental element of length $d\ell_\theta$ which is associated with the change of the angle θ by $d\theta$ is $d\ell_\theta = r d\theta$, whereas the element $d\ell_\phi$ associated with the change of the angle ϕ by $d\phi$ is given by $d\ell_\phi = r \sin\theta \, d\phi$. From Figure 1.17, it is clear that the metric coefficient $r \sin\theta$ is basically the projection of r in the x-y plane where the incremental change in the angle ϕ occurs. $d\ell_\phi$ is therefore obtained from the relation

$$d\ell_\phi = \text{projection of } r \text{ in the } x\text{-}y \text{ plane} \times d\phi$$
$$= r \sin\theta \, d\phi$$

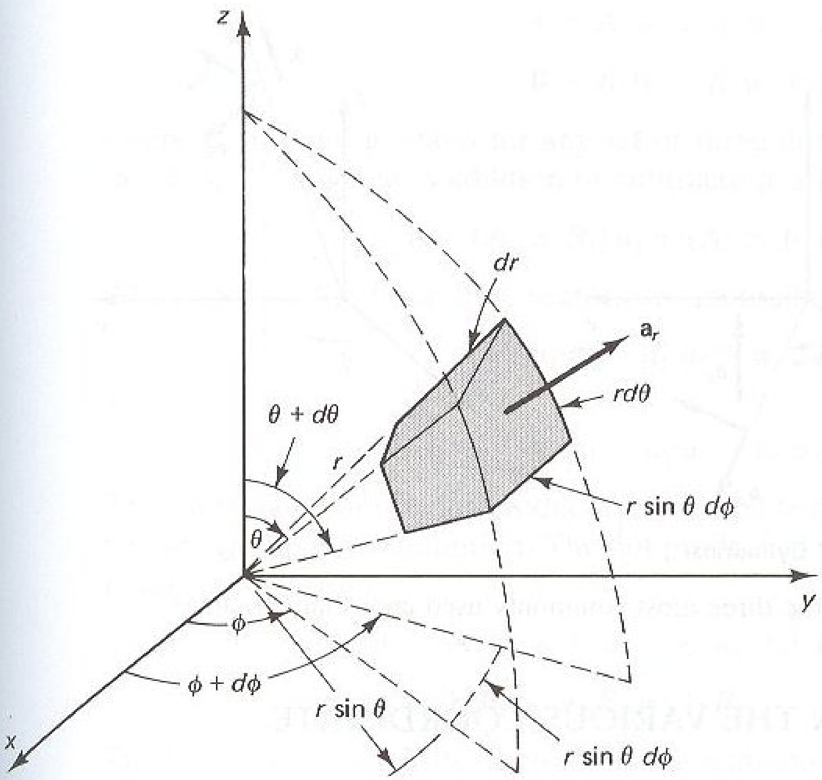

Figure 1.17 The elements of length, surface, and volume in the spherical coordinate system.

Based on the preceding discussion and from Figure 1.17, it is fairly straightforward to show that the incremental element of volume dv is given by

$$dv = dr(rd\theta)(r \sin\theta\, d\phi)$$

$$= r^2 \sin\theta\, dr\, d\theta\, d\phi$$

The element of length $d\ell$ from P_1 to P_2 is

$$d\ell = dr\, \mathbf{a}_r + d\ell_\theta\, \mathbf{a}_\theta + d\ell_\phi\, \mathbf{a}_\phi$$

$$= dr\, \mathbf{a}_r + rd\theta\, \mathbf{a}_\theta + r \sin\theta\, d\phi\, \mathbf{a}_\phi$$

The various elements of area are given by

$$ds_r = r^2 \sin\theta\, d\theta\, d\phi\, \mathbf{a}_r$$

$$ds_\theta = r \sin\theta\, dr\, d\phi\, \mathbf{a}_\theta$$

$$ds_\phi = r\, dr\, d\theta\, \mathbf{a}_\phi$$

Clearly each element of area is associated with a unit vector perpendicular to it. In Figure 1.17, the unit vector $\mathbf{a}_r$ of the element of area ds_r is indicated.

A summary of the base vectors in the Cartesian, cylindrical, and the spherical coordinate systems is given in Figure 1.18. It should be noted that the base vectors in the spherical coordinate system are similar to those in the cylindrical coordinates insofar as they change their directions at various points in the coordinate system.

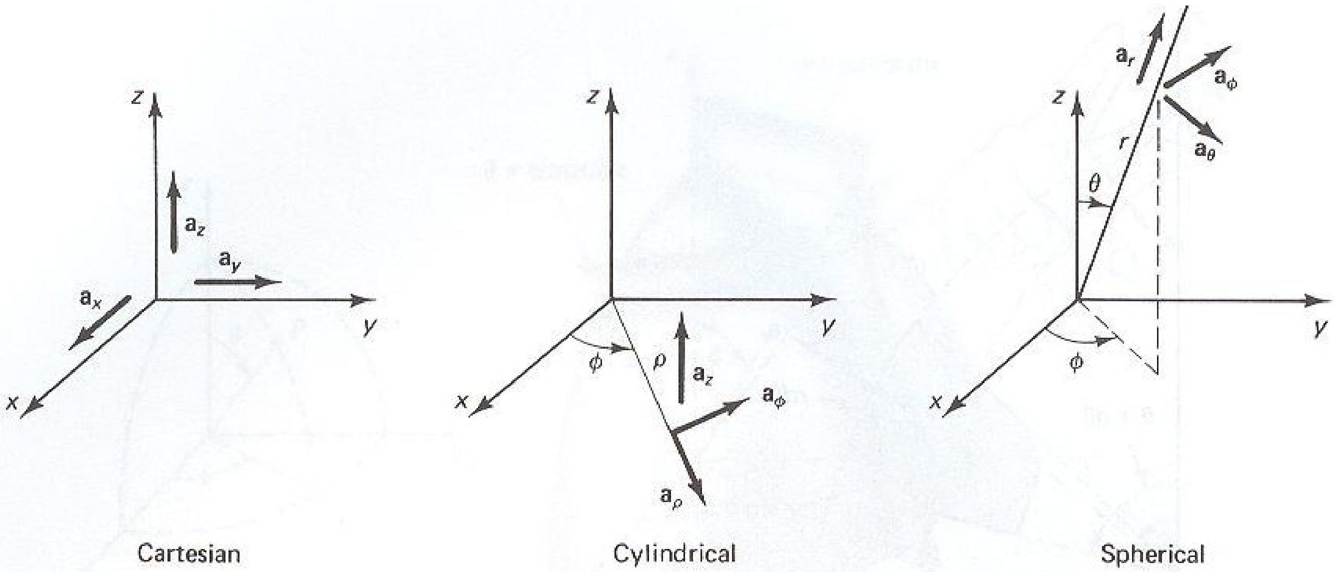

Figure 1.18 The base vectors of the three most commonly used coordinate systems.

1.4 VECTOR REPRESENTATION IN THE VARIOUS COORDINATE SYSTEMS

A vector quantity is completely specified in any coordinate system if the origin of the vector and its components (projections) in the directions of the three base vectors are known. For example, components of a vector **A** are designated by A_x, A_y, A_z in the Cartesian coordinate system, by A_ρ, A_ϕ, A_z in the cylindrical coordinate system, and by A_r, A_θ, A_ϕ in the spherical coordinate system. The vector **A** may then be represented in terms of its components as:

$$\mathbf{A} = A_x\,\mathbf{a}_x + A_y\,\mathbf{a}_y + A_z\,\mathbf{a}_z \text{ (Cartesian system)}$$

$$\mathbf{A} = A_\rho\,\mathbf{a}_\rho + A_\phi\,\mathbf{a}_\phi + A_z\,\mathbf{a}_z \text{ (Cylindrical system)}$$

$$\mathbf{A} = A_r\,\mathbf{a}_r + A_\theta\,\mathbf{a}_\theta + A_\phi\,\mathbf{a}_\phi \text{ (Spherical system)}$$

Let us now consider two vectors **A** and **B** that have origins at the same point in any one of these coordinate systems. It is important to note that the unit vectors are directed in the same directions at all points only in the Cartesian coordinate system. We illustrated in the previous sections that in the cylindrical and the spherical coordinate systems the unit vectors generally have different directions at different points. Therefore, in all the vector operations that we will describe in this section, it will be assumed that either the vectors are *originating from the same point* in the coordinate system and are thus expressed in terms of the same base vectors, or that the vectors are originating at different points and their components are all expressed in terms of a single set of the base vectors at either one of the two origins of the two vectors. What is important here is that the two vectors are expressed in terms of their components along the *same base vectors*. Let us now consider two vectors, **A** and **B**, expressed in terms of the same base vector, $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$.

$$\mathbf{A} = A_1\mathbf{u}_1 + A_2\mathbf{u}_2 + A_3\mathbf{u}_3$$

$$\mathbf{B} = B_1\mathbf{u}_1 + B_2\mathbf{u}_2 + B_3\mathbf{u}_3$$

where $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ stand for any set of three unit vectors $(\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z)$, $(\mathbf{a}_\rho, \mathbf{a}_\phi, \mathbf{a}_z)$, or $(\mathbf{a}_r, \mathbf{a}_\theta, \mathbf{a}_\phi)$. The vector's addition or subtraction is given by

$$\mathbf{A} \pm \mathbf{B} = (A_1 \pm B_1)\mathbf{u}_1 + (A_2 \pm B_2)\mathbf{u}_2 + (A_3 \pm B_3)\mathbf{u}_3$$

Also, because the three base vectors are mutually orthogonal, therefore

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = \mathbf{u}_1 \cdot \mathbf{u}_3 = \mathbf{u}_2 \cdot \mathbf{u}_3 = 0$$

and

$$\mathbf{u}_1 \cdot \mathbf{u}_1 = \mathbf{u}_2 \cdot \mathbf{u}_2 = \mathbf{u}_3 \cdot \mathbf{u}_3 = 1$$

The unity value in the dot product is indicated because the magnitudes of these base vectors are unity by definition. The dot product of two vectors *with origins at the same points* is, therefore,

$$\mathbf{A} \cdot \mathbf{B} = (A_1\mathbf{u}_1 + A_2\mathbf{u}_2 + A_3\mathbf{u}_3) \cdot (B_1\mathbf{u}_1 + B_2\mathbf{u}_2 + B_3\mathbf{u}_3)$$
$$= A_1B_1 + A_2B_2 + A_3B_3$$

Furthermore, because the unit vectors are mutually orthogonal, we have the following relations for the cross products

$$\mathbf{u}_1 \times \mathbf{u}_2 = \mathbf{u}_3, \qquad \mathbf{u}_2 \times \mathbf{u}_3 = \mathbf{u}_1, \qquad \mathbf{u}_3 \times \mathbf{u}_1 = \mathbf{u}_2$$

and

$$\mathbf{u}_1 \times \mathbf{u}_1 = \mathbf{u}_2 \times \mathbf{u}_2 = \mathbf{u}_3 \times \mathbf{u}_3 = 0$$

The cross product of two $\mathbf{A}$ and $\mathbf{B}$ vectors may then be expressed in the form

$$\mathbf{A} \times \mathbf{B} = (A_1\mathbf{u}_1 + A_2\mathbf{u}_2 + A_3\mathbf{u}_3) \times (B_1\mathbf{u}_1 + B_2\mathbf{u}_2 + B_3\mathbf{u}_3)$$
$$= \mathbf{u}_1(A_2B_3 - A_3B_2) + \mathbf{u}_2(A_3B_1 - A_1B_3) + \mathbf{u}_3(A_1B_2 - A_2B_1)$$

which can be written in the form of a determinant:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

which is an easier form to remember.

EXAMPLE 1.1

Which of the following sets of independent variables (coordinates) define a point in a coordinate system?

1. $x = 2, y = -4, z = 0$.
2. $\rho = -4, \phi = 0°, z = -1$.
3. $r = 3, \theta = -90°, \phi = 0°$.

Solution

Only the point in (a), because ρ in (b) and θ in (c) have to be positive, that is, $\rho \geqq 0$ and $0 \leqq \theta \leqq \pi$, which they are not.

EXAMPLE 1.2

Find a unit vector normal to the plane containing the following two vectors:

$$\mathbf{OA} = 4\,\mathbf{a}_x + 10\,\mathbf{a}_y$$

$$\mathbf{OB} = 4\,\mathbf{a}_x + 5\,\mathbf{a}_z$$

Solution

The cross product of two vectors **OA** and **OB** is a vector quantity whose magnitude is equal to the product of the magnitudes of **OA** and **OB** and the sine of the angle between them, and whose direction is perpendicular to the plane containing the two vectors. Hence,

$$\mathbf{OA} \times \mathbf{OB} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 4 & 10 & 0 \\ 4 & 0 & 5 \end{vmatrix} = 50\,\mathbf{a}_x - 20\,\mathbf{a}_y - 40\,\mathbf{a}_z$$

The required unit vector is obtained by dividing **OA** $\times$ **OB** by its magnitude; hence,

$$\mathbf{a}_n = \frac{50\mathbf{a}_x - 20\mathbf{a}_y - 40\mathbf{a}_z}{|50\mathbf{a}_x - 20\mathbf{a}_y - 40\mathbf{a}_z|} = \frac{5\mathbf{a}_x - 2\mathbf{a}_y - 4\mathbf{a}_z}{\sqrt{25 + 4 + 16}}$$

$$= \frac{1}{3\sqrt{5}}(5\mathbf{a}_x - 2\mathbf{a}_y - 4\mathbf{a}_z)$$

EXAMPLE 1.3

Show that vectors $\mathbf{A} = \mathbf{a}_x + 4\mathbf{a}_y + 3\mathbf{a}_z$ and $\mathbf{B} = 2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z$ are perpendicular to each other.

Solution

The dot product consists of multiplying the magnitude of one vector by the projection of the second along the direction of the first. The dot product of two perpendicular vectors is therefore zero. For the two vectors given in this example,

$$\mathbf{A}\cdot\mathbf{B} = 2 + 4 - 6 = 0$$

so that **A** and **B** are perpendicular.

EXAMPLE 1.4

The two vectors **A** and **B** are given by

$$\mathbf{A} = \mathbf{a}_\rho + \pi\,\mathbf{a}_\phi + 3\,\mathbf{a}_z$$

$$\mathbf{B} = \alpha\,\mathbf{a}_\rho + \beta\,\mathbf{a}_\phi - 6\,\mathbf{a}_z$$

Determine α and β such that the two vectors are parallel.

Solution

For these two vectors to be parallel the cross product of **A** and **B** should be zero, that is,

$$\mathbf{A} \times \mathbf{B} = 0$$

$$= \begin{vmatrix} \mathbf{a}_\rho & \mathbf{a}_\phi & \mathbf{a}_z \\ 1 & \pi & 3 \\ \alpha & \beta & -6 \end{vmatrix}$$

$$0 = \mathbf{a}_\rho(-6\pi - 3\beta) + \mathbf{a}_\phi(3\alpha + 6) + \mathbf{a}_z(\beta - \pi\alpha)$$

For the vector that resulted from the cross product to be zero, each one of its components should be independently zero. Hence,

$$-6\pi - 3\beta = 0, \qquad \therefore \beta = -2\pi$$

and

$$3\alpha + 6 = 0, \qquad \therefore \alpha = -2$$

These two values of α and β clearly satisfy the remaining relation $\beta - \pi\alpha = 0$. The vector **B** is therefore given by

$$\mathbf{B} = -2\,\mathbf{a}_\rho - 2\pi\,\mathbf{a}_\phi - 6\,\mathbf{a}_z$$

1.5 VECTOR COORDINATE TRANSFORMATION

The vector coordinate transformation is basically a process in which we change a vector representation from one coordinate system to another. This procedure is similar to scalar coordinate transformation with the additional necessity of transforming the individual components of the vector from being along the base vectors of the first coordinate system to components along the base vectors of the other coordinate system. Therefore, the transformation of a vector representation from one coordinate system to another involves a two-step process which includes the following:

a. *Changing the independent variables* (e.g., expressing x, y, and z of the rectangular coordinate system in terms of ρ, φ, and z of the cylindrical coordinate system or r, θ, φ of the spherical coordinate system).

b. *Changing the components of the vector* from those along the unit vectors of one coordinate system to those along the unit vectors of the other (e.g., changing the components from those along $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ in the Cartesian coordinate system to components along $\mathbf{a}_\rho$, $\mathbf{a}_\phi$, and $\mathbf{a}_z$ in the cylindrical coordinate system).

In the following sections we shall describe specific transformation of the independent variables and the vector components from one coordinate system to another.

1.5.1 Cartesian-to-Cylindrical Transformation

The relation between the independent variables of these two coordinate systems is shown in Figure 1.19a. From Figure 1.19a it may be seen that

$$x = \rho \cos\phi \qquad \rho = \sqrt{x^2 + y^2}$$
$$y = \rho \sin\phi \qquad \phi = \tan^{-1}(y/x)$$
$$z = z \text{ (the same in both coordinates)}$$

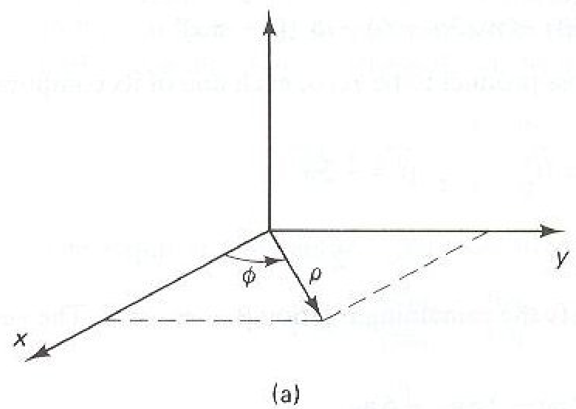

Figure 1.19a Relation between the independent variables in the Cartesian and cylindrical coordinate systems.

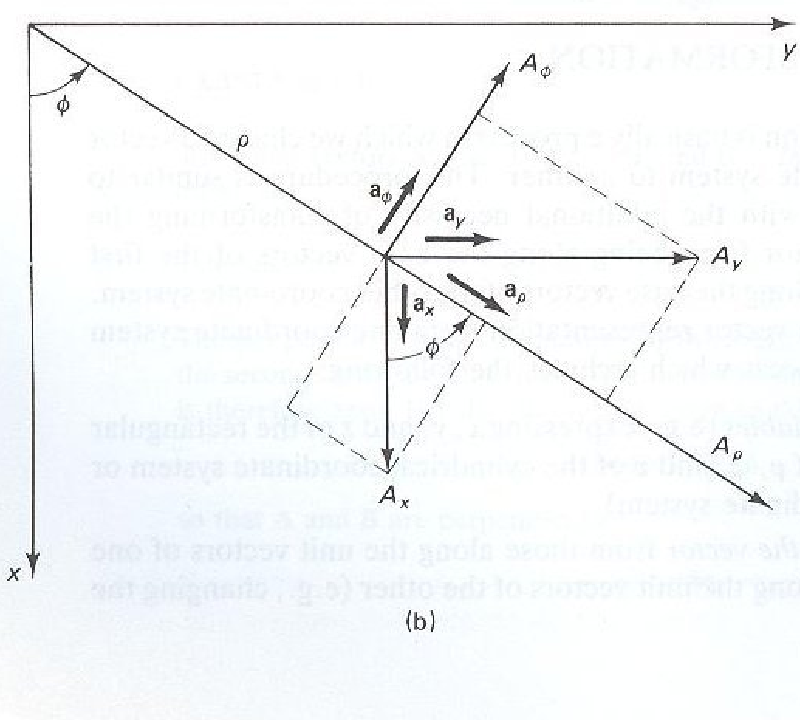

Figure 1.19b The relation between the vector components in the rectangular and cylindrical coordinate systems.

To illustrate the process of changing the components of a vector from being along the base vectors of one coordinate system (e.g., the Cartesian) to the other components along the base vectors of the other coordinate system (e.g., the cylindrical), let us solve the following example.

EXAMPLE 1.5

Transform the vector **A** given in the Cartesian coordinate system by

$$\mathbf{A} = A_x\,\mathbf{a}_x + A_y\,\mathbf{a}_y + A_z\,\mathbf{a}_z$$

to the form

$$\mathbf{A} = A_\rho\,\mathbf{a}_\rho + A_\phi\,\mathbf{a}_\phi + A_z\,\mathbf{a}_z$$

in the cylindrical coordinate system.

Solution

In transforming the vector **A** from the Cartesian coordinate system to the cylindrical one, it is required to obtain the components A_ρ, A_ϕ, and A_z of the vector **A** along the base vectors $\mathbf{a}_\rho$, $\mathbf{a}_\phi$, and $\mathbf{a}_z$ in the cylindrical coordinate systems. From the definition of the dot product, the component of **A** along $\mathbf{a}_\rho$ is given by

$$A_\rho = \mathbf{A}\cdot\mathbf{a}_\rho = (A_x\,\mathbf{a}_x + A_y\,\mathbf{a}_y + A_z\,\mathbf{a}_z)\cdot\mathbf{a}_\rho$$

$$= A_x\,\mathbf{a}_x\cdot\mathbf{a}_\rho + A_y\,\mathbf{a}_y\cdot\mathbf{a}_\rho + A_z\,\mathbf{a}_z\cdot\mathbf{a}_\rho$$

$\mathbf{a}_x\cdot\mathbf{a}_\rho$ from Figure 1.19b is equal to $\cos\phi$ because the magnitudes of both $\mathbf{a}_x$ and $\mathbf{a}_\rho$ are both equal to unity and the angle between them is ϕ. Similarly, $\mathbf{a}_y\cdot\mathbf{a}_\rho = \cos(\pi/2 - \phi) = \sin\phi$, and $\mathbf{a}_z\cdot\mathbf{a}_\rho = 0$. A_ρ is therefore given by

$$A_\rho = A_x\cos\phi + A_y\sin\phi$$

which is the same result previously obtained using the projections of the vector components. Similarly, the A_ϕ component may be obtained by

$$A_\phi = \mathbf{A}\cdot\mathbf{a}_\phi = A_x\,\mathbf{a}_x\cdot\mathbf{a}_\phi + A_y\,\mathbf{a}_y\cdot\mathbf{a}_\phi + A_z\,\mathbf{a}_z\cdot\mathbf{a}_\phi$$

$$= -A_x\sin\phi + A_y\cos\phi$$

The negative sign of the A_x component is included because the component $A_x\sin\phi$ is not along the positive $\mathbf{a}_\phi$ direction but instead along the negative $\mathbf{a}_\phi$ direction. Alternatively, the negative sign may be considered as a result of the fact that the angle between $\mathbf{a}_x$ and $\mathbf{a}_\phi$ is $(\pi/2 + \phi)$. The dot product $\mathbf{a}_x\cdot\mathbf{a}_\phi$ requires calculation of the cosine of the angle between them and $\cos(\pi/2 + \phi) = -\sin\phi$.

The A_z component of the vector will, of course, remain unchanged between the Cartesian and cylindrical coordinate systems.

1.5.2 Cartesian-to-Spherical Transformation

The relations between the independent variables can be obtained from Figure 1.20. It should be noted that $r_{\|}$ in Figure 1.20 is simply the projection of r in the x-y plane and

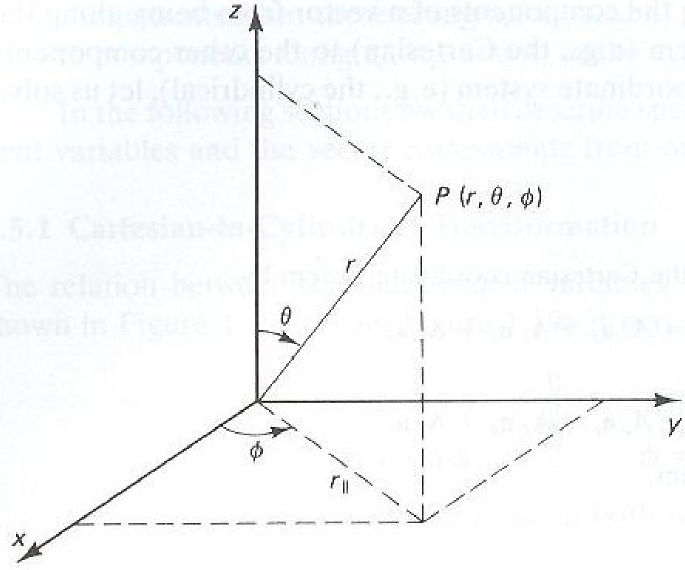

Figure 1.20 Relation between the independent variables in the spherical and Cartesian coordinate systems.

hence is given by $r_\| = r \sin\theta$. Once again, to illustrate expressing the vector components from one coordinate system to another, we will solve the following example.

EXAMPLE 1.6

Derive the vector components transformation from the Cartesian to the spherical coordinate systems and vice versa.

$$x = r_\| \cos\phi \qquad r = \sqrt{x^2 + y^2 + z^2}$$
$$= r \sin\theta \cos\phi \qquad \phi = \tan^{-1}(y/x)$$
$$y = r_\| \sin\phi \qquad \theta = \tan^{-1}\sqrt{(x^2 + y^2)}/z$$
$$= r \sin\theta \sin\phi$$
$$z = r \cos\theta$$

Solution

The problem can be alternatively stated by considering the vector **A**, which is given in the Cartesian coordinate system $\mathbf{A} = A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z$, and it is required to find the vector components A_r, A_θ, and A_ϕ along the $\mathbf{a}_r$, $\mathbf{a}_\theta$, and $\mathbf{a}_\phi$ unit vectors in the spherical coordinate system. The relationship between the vector components is illustrated in Figure 1.21.

From Figure 1.21, it may be seen that the projections of the components A_x, A_y, and A_z along the direction $\mathbf{a}_r$ are given, respectively, by $\cos\phi \sin\theta$, $\sin\phi \sin\theta$, and $\cos\theta$. Therefore, the radial component A_r of the vector **A** is given by

$$A_r = A_x \cos\phi \sin\theta + A_y \sin\phi \sin\theta + A_z \cos\theta \tag{1.1}$$

Following a similar procedure, we next find the projections of A_x, A_y, and A_z in the directions of $\mathbf{a}_\theta$ and $\mathbf{a}_\phi$. These are given, respectively, by $(\cos\phi \cos\theta, \sin\phi \cos\theta, -\sin\theta)$ along the $\mathbf{a}_\theta$ direction, and $(-\sin\phi, \cos\phi)$ along the $\mathbf{a}_\phi$ direction. Hence,

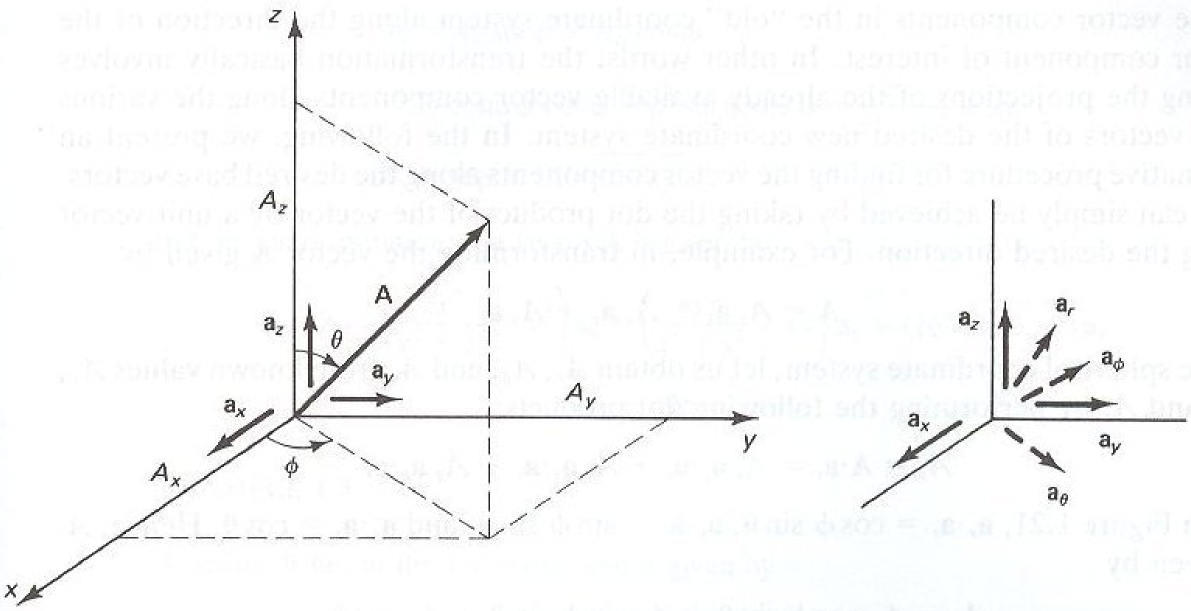

Figure 1.21 Transformation of the vector components from the Cartesian to the spherical coordinate system.

$$A_\theta = A_x \cos\phi \cos\theta + A_y \sin\phi \cos\theta - A_z \sin\theta \quad (1.2)$$

$$A_\phi = -A_x \sin\phi + A_y \cos\phi \quad (1.3)$$

and the vector **A** expressed in the spherical coordinates system is given by

$$\mathbf{A} = A_r \mathbf{a}_r + A_\theta \mathbf{a}_\theta + A_\phi \mathbf{a}_\phi$$

where the components A_r, A_θ, and A_ϕ are given in equations 1.1 to 1.3.

To find the inverse transformation, we simply start with the vector **A** given in the spherical coordinate system by

$$\mathbf{A} = A_r \mathbf{a}_r + A_\theta \mathbf{a}_\theta + A_\phi \mathbf{a}_\phi$$

and then find the components of A_r, A_θ, and A_ϕ along the unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ of the Cartesian coordinate system. Alternatively, we can just solve the set of equations 1.1 to 1.3 simultaneously for A_x, A_y, and A_z. The result in both cases is

$$A_x = A_r \sin\theta \cos\phi + A_\theta \cos\theta \cos\phi - A_\phi \sin\phi \quad (1.4)$$

$$A_y = A_r \sin\theta \sin\phi + A_\theta \cos\theta \sin\phi + A_\phi \cos\phi \quad (1.5)$$

and

$$A_z = A_r \cos\theta - A_\theta \sin\theta \quad (1.6)$$

Clearly, the vector **A** in the Cartesian coordinate system is given in terms of its components A_x, A_y, and A_z given in equations 1.4 to 1.6.

Alternative Procedure. In the previous sections we described a process for making the vector coordinate transformation by dealing with each of the vector components in the "new" coordinate system and deriving expressions for the contributions of the vector components in the "old" coordinate system along the direction of the vector component of interest. In other words, the transformation basically involves finding the projections of the already available vector components along the various base vectors of the desired new coordinate system. In the following, we present an alternative procedure for finding the vector components along the desired base vectors. This can simply be achieved by taking the dot product of the vector by a unit vector along the desired direction. For example, in transforming the vector **A** given by

$$\mathbf{A} = A_x\,\mathbf{a}_x + A_y\,\mathbf{a}_y + A_z\,\mathbf{a}_z$$

to the spherical coordinate system, let us obtain A_r, A_θ, and A_ϕ from known values A_x, A_y, and A_z by performing the following dot products:

$$A_r = \mathbf{A}\cdot\mathbf{a}_r = A_x\,\mathbf{a}_x\cdot\mathbf{a}_r + A_y\,\mathbf{a}_y\cdot\mathbf{a}_r + A_z\,\mathbf{a}_z\cdot\mathbf{a}_r$$

From Figure 1.21, $\mathbf{a}_x\cdot\mathbf{a}_r = \cos\phi\,\sin\theta$, $\mathbf{a}_y\cdot\mathbf{a}_r = \sin\phi\,\sin\theta$, and $\mathbf{a}_z\cdot\mathbf{a}_r = \cos\theta$. Hence, A_r is given by

$$A_r = A_x\,\cos\phi\,\sin\theta + A_y\,\sin\phi\,\sin\theta + A_z\,\cos\theta$$

which is the same result we obtained in the previous section. Similarly, it can be shown that

$$A_\theta = \mathbf{A}\cdot\mathbf{a}_\theta = A_x\,\mathbf{a}_x\cdot\mathbf{a}_\theta + A_y\,\mathbf{a}_y\cdot\mathbf{a}_\theta + A_z\,\mathbf{a}_z\cdot\mathbf{a}_\theta$$

Once again from Figure 1.21, it is quite clear that $\mathbf{a}_x\cdot\mathbf{a}_\theta = \cos\phi\,\cos\theta$, $\mathbf{a}_y\cdot\mathbf{a}_\theta = \sin\phi\,\cos\theta$, and $\mathbf{a}_z\cdot\mathbf{a}_\theta = -\sin\theta$. Hence,

$$A_\theta = A_x\,\cos\phi\,\cos\theta + A_y\,\sin\phi\,\cos\theta - A_z\,\sin\theta$$

EXAMPLE 1.7

Express the vector

$$\mathbf{A} = z\,\cos\phi\,\mathbf{a}_\rho + \rho^2\,\sin\phi\,\mathbf{a}_\phi + 16\rho\,\mathbf{a}_z$$

in the Cartesian coordinates.

Solution

We first change the independent variables from ρ, ϕ, and z in the cylindrical coordinate system to x, y, and z in the Cartesian coordinate system. These changes are previously indicated as

$$\rho = \sqrt{x^2 + y^2}, \qquad \cos\phi = \frac{x}{\sqrt{x^2 + y^2}}, \qquad \sin\phi = \frac{y}{\sqrt{x^2 + y^2}}$$

Next we use the vector component transformation between the two coordinate systems. From the relations given in example 1.5, we obtain

$$\begin{aligned} A_x &= A_\rho\cos\phi - A_\phi\sin\phi \\ &= z\,\cos^2\phi - \rho^2\,\sin^2\phi = \frac{zx^2}{x^2 + y^2} - y^2 \end{aligned}$$

$$A_y = A_\rho \sin\phi + A_\phi \cos\phi$$

$$= z \sin\phi \cos\phi + \rho^2 \sin\phi \cos\phi = \frac{zxy}{x^2 + y^2} + xy$$

$$A_z = 16\rho = 16\sqrt{x^2 + y^2}$$

and, in vector notation, the vector **A** is given by:

$$\mathbf{A} = \left(\frac{zx^2}{x^2 + y^2} - y^2\right)\mathbf{a}_x + \left(\frac{xyz}{x^2 + y^2} + xy\right)\mathbf{a}_y + \left(16\sqrt{x^2 + y^2}\right)\mathbf{a}_z$$

EXAMPLE 1.8

A vector **B** lies in the *x*-*y* plane, and is given by

$$\mathbf{B} = x\,\mathbf{a}_x + y\,\mathbf{a}_y$$

1. Obtain an expression for **B** in cylindrical coordinates.
2. Determine the magnitude and direction of **B** at the point $x = 3, y = 4$.

Solution

1. Using the coordinate transformation

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi$$

and the vector transformation given in example 1.5, we obtain

$$B_\rho = B_x \cos\phi + B_y \sin\phi = x\cos\phi + y\sin\phi$$

$$= \rho\cos^2\phi + \rho\sin^2\phi = \rho$$

$$B_\phi = -B_x \sin\phi + B_y \cos\phi$$

$$= -\rho\sin\phi\cos\phi + \rho\sin\phi\cos\phi = 0$$

$$B_z = 0$$

and, in vector notation,

$$\mathbf{B} = \rho\,\mathbf{a}_\rho$$

2. At the point $x = 3, y = 4$, the radial distance is

$$\rho = x^2 + y^2 = 5$$

Hence,

$$\mathbf{B} = 5\,\mathbf{a}_\rho$$

EXAMPLE 1.9

Express the vector $\mathbf{A} = \frac{x^2 z}{y}\mathbf{a}_x$ in the spherical coordinate system.

Solution

Because the vector **A** has only an A_x component, its component A_r along the base vectors $\mathbf{a}_r$ in the spherical coordinate system is given by

$$A_r = A_x \cos\phi \sin\theta$$

$$A_r = \frac{x^2 z}{y}\cos\phi\sin\theta = \frac{(r\sin\theta\cos\phi)^2 r\cos\theta}{r\sin\theta\sin\phi}\cos\phi\sin\theta$$

$$= r^2\frac{\sin^2\theta\cos\theta\cos^3\phi}{\sin\phi}$$

Similarly, the A_θ and A_ϕ components are given by

$$A_\theta = A_x\cos\phi\cos\theta = \frac{x^2 z}{y}\cos\phi\cos\theta$$

$$= \frac{(r^2\sin^2\theta\cos^2\phi)(r\cos\theta)\cos\phi\cos\theta}{r\sin\theta\sin\phi}$$

$$= r^2\frac{\sin\theta\cos^2\theta\cos^3\phi}{\sin\phi}$$

$$A_\phi = -A_x\sin\phi = -\frac{(r^2\sin^2\theta\cos^2\phi)(r\cos\theta)}{r\sin\theta\sin\phi}\sin\phi$$

$$= -r^2\sin\theta\cos\theta\cos^2\phi$$

It is rather surprising to see that a simple vector such as **A** that has only one A_x component in the Cartesian coordinate system actually has three components of complicated expressions in the spherical coordinate system

$$\mathbf{A} = A_r\mathbf{a}_r + A_\theta\mathbf{a}_\theta + A_\phi\mathbf{a}_\phi$$

This problem emphasizes the importance of choosing the right coordinate system that best fits the representation of a given vector.

1.6 ELECTRIC AND MAGNETIC FIELDS

Basic to our study of electromagnetics is an understanding of the concept of electric and magnetic fields. Before studying electromagnetic fields, however, we must first define what is meant by a field. A field is associated with a region in space, and we say that a field exists in the region if there is a physical phenomenon associated with points in that region. In other words, we can talk of the field of any physical quantity as being a description of how the quantity varies from one point to another in the region of the field. For example, we are familiar with the earth's gravitational field; we do not "see" the field, but we know of its existence in the sense that objects of given mass are acted on by the gravitational force of the earth.

1.6.1 Coulomb's Law and Electric Field Intensity

We are all familiar with Newton's law of universal gravitation, which states that every object of mass m in the universe attracts every other object m' with a force that is directly proportional to the product of the masses and inversely proportional to the square of the distance R between them—that is,

$$\mathbf{F} = G\frac{mm'}{R^2}\mathbf{a}$$

where G is the gravitational constant and $\mathbf{a}$ is a unit vector along the straight line joining the two masses. The equation above simply means that there is a gravitational force of attraction between bodies of given masses and that this force is along the line joining the two masses. In a similar manner, a force field known as the *electric field* is associated with bodies that are *charged*.

In the experiments conducted by Coulomb, he showed that for two charged bodies that are very small in size compared with their separation—so that they may be considered as *point charges*—the following hold:

1. The magnitude of the force is proportional to the product of the magnitudes of the charges.
2. The magnitude of the force is inversely proportional to the square of the distance between the charges.
3. The direction of the force is along the line joining the charges.
4. The magnitude of the force depends on the medium.
5. Like charges repel; unlike charges attract.

Hence, if we consider two point charges Q_1 and Q_2 separated by a distance R, the force is then given by:

$$\boxed{\mathbf{F} = k\frac{Q_1Q_2}{R^2}\mathbf{a}_{12}}$$

where k is a proportionality constant and $\mathbf{a}_{12}$ is a unit vector along the line joining the two charges as indicated by the third observation in the experiment by Coulomb. If the international system of units (SI system) is used, then Q is measured in coulombs (C), R in meters (m), and the force should be in newtons (N) (see Appendix B). In this case, the constant of proportionality k will be

$$k = \frac{1}{4\pi\epsilon_o}$$

where ϵ_o is called the permittivity of air (vacuum) and has a value measured in farads per meter (F/m),

$$\epsilon_0 = 8.854 \times 10^{-12} \approx \frac{1}{36\pi} \times 10^{-9} \text{ F/m}$$

The direction of the force in the above equation should actually be defined in terms of two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ experienced by Q_1 and Q_2, respectively. These two forces with their appropriate directions are given by

$$\mathbf{F}_1 = \frac{Q_1 Q_2}{4\pi\epsilon_o R^2} \mathbf{a}_{21}$$

$$\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_o R^2} \mathbf{a}_{12}$$

where $\mathbf{a}_{21}$ and $\mathbf{a}_{12}$ are unit vectors along the line joining Q_1 and Q_2 as shown in Figure 1.22.

Electric Field Intensity. From Coulomb's law, if we let one of the two charges, say Q_2, be a small test charge q, we have

$$\mathbf{F}_2 = \frac{Q_1 q}{4\pi\epsilon_o R^2} \mathbf{a}_{12}$$

The electric field intensity $\mathbf{E}_2$ at the location of the test charge owing to the point charge Q_1 is defined as

$$\mathbf{E}_2 = \frac{\mathbf{F}_2}{q} = \frac{Q_1}{4\pi\epsilon_o R^2} \mathbf{a}_{12}$$

In general, the electric field intensity **E** *is defined as the vector force on a unit positive test charge.*

$$\boxed{\mathbf{E} = \frac{Q}{4\pi\epsilon_o R^2} \mathbf{a}_R}$$

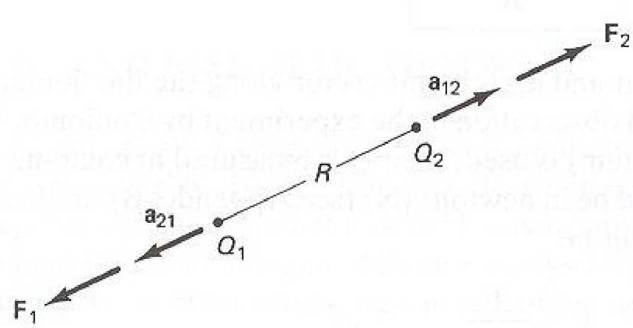

Figure 1.22 The electric force between two point charges Q_1 and Q_2.

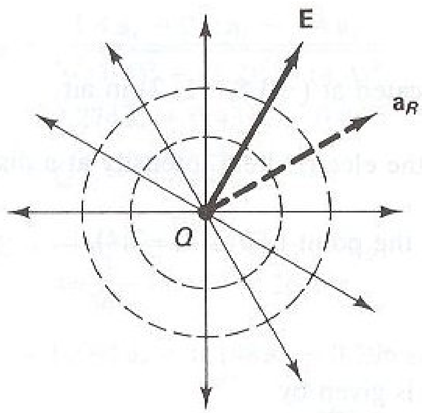

Figure 1.23 Direction lines and constant-magnitude surfaces of electric field owing to a point charge.

where $\mathbf{a}_R$ is a unit vector along the line joining the point charge Q and the test point wherever it is (the test point in this case is the point at which the value of the electric field intensity E is desired). The electric field intensity owing to a positive point charge is thus directed everywhere radially away from the point charge, and its constant magnitude surfaces are spherical surfaces centered at the point charge as shown in Figure 1.23.

If we have N point charges $Q_1, Q_2, \ldots, Q_N$ as shown in Figure 1.24, the force experienced by a test charger q placed at a point P is the vector sum of the forces experienced by the test charge owing to the individual charges, that is,

$$\boxed{\mathbf{E} = \sum_{i=1}^{N} \frac{Q_i}{4\pi\epsilon_o R_i^2} \mathbf{a}_{R_i}}$$

and

$$\mathbf{F} = q\mathbf{E}$$

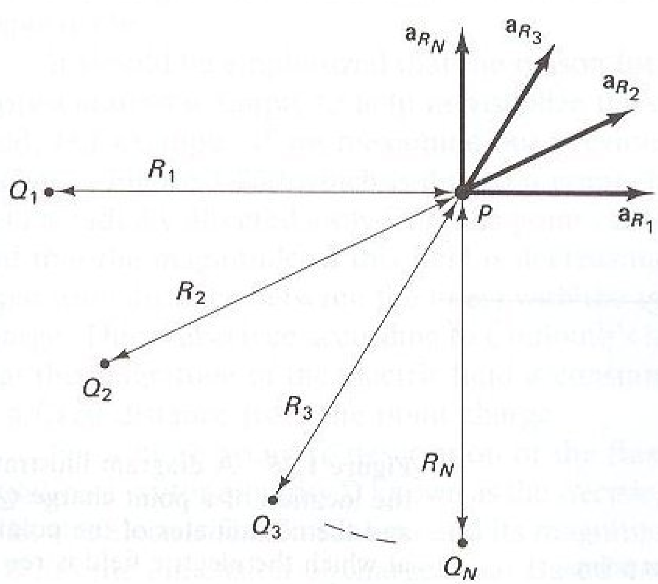

Figure 1.24 The total electric field intensity at point P owing to N point charges equals the vector sum of the electric field intensities owing to all of the charges.

EXAMPLE 1.10

A point charge $Q = 10^{-9}$ C is located at $(-0.5, -1, 2)$ in air.

1. What is the magnitude of the electric field intensity at a distance of 1 m from the charge?
2. Find the electric field **E** at the point $(0.9, 1.2, -2.4)$.

Solution

1. The electric field intensity is given by

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_o R^2}\,\mathbf{a}_R$$

which is in the radial direction, and its magnitude $|\mathbf{E}|$ at $R = 1$ m is given by

$$|\mathbf{E}| = \frac{10^{-9}}{4\pi\dfrac{1}{36\pi}\times 10^{-9}} = 9 \text{ N/C}$$

2. A diagram illustrating the locations of the charge and the test point is shown in Figure 1.25. **E** at $(0.9, 1.2, -2.4)$ is

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_o (QT)^2}\,\mathbf{a}_{QT}$$

$$\begin{aligned}\mathbf{QT} &= \mathbf{OT} - \mathbf{OQ}\\ &= (0.9 - (-0.5))\,\mathbf{a}_x + (1.2 - (-1))\,\mathbf{a}_y + (-2.4 - (2))\,\mathbf{a}_z\\ &= 1.4\,\mathbf{a}_x + 2.2\,\mathbf{a}_y - 4.4\,\mathbf{a}_z\end{aligned}$$

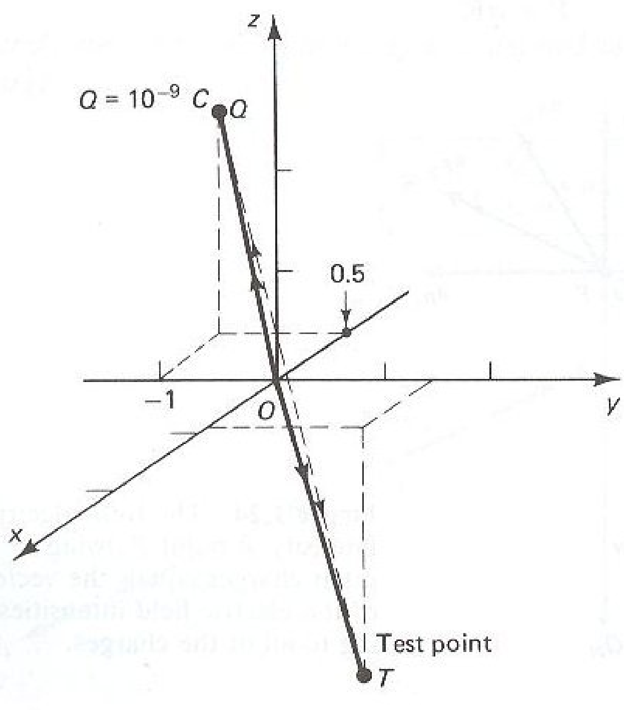

Figure 1.25 A diagram illustrating the location of a point charge Q and the coordinates of the point T at which the electric field is required.

$$\mathbf{a}_{QT} = \frac{1.4\,\mathbf{a}_x + 2.2\,\mathbf{a}_y - 4.4\,\mathbf{a}_z}{\sqrt{(1.4)^2 + (2.2)^2 + (4.4)^2}}$$
$$= 0.274\,\mathbf{a}_x + 0.43\,\mathbf{a}_y - 0.86\,\mathbf{a}_z$$
$$|\mathbf{QT}| = QT = \sqrt{26.16}$$
$$\mathbf{E} = \frac{10^{-9}}{4\pi\frac{1}{36\pi} \times 10^{-9} \times 26.16}\,\mathbf{a}_{QT}$$
$$= 0.094\,\mathbf{a}_x + 0.148\,\mathbf{a}_y - 0.296\,\mathbf{a}_z \text{ N/C}$$

1.6.2 Flux Representation of Vector Field

As indicated in earlier sections, a vector quantity is completely specified in terms of its magnitude and direction. Therefore, the variation of a vector field in space can be graphically illustrated by drawing different vectors at various points in the field region as shown in Figure 1.26a. The magnitudes and the directions of these vectors represent the different values of the field (magnitude and direction) at the various points in space. Although the graphical representation in Figure 1.26a is possible and correct, it is a rather poor illustration and might get confusing for fields with rapid spatial variation. A widely adopted graphical representation of vector fields is in terms of their flux lines. In this procedure, a vector field is represented by arrows of the same length but of different separation between them. The direction of these arrows (flux lines) is in the direction of the vector field (or tangential to it). The magnitude of the field in this case, however, is not described in terms of the length of the arrow but instead in terms of the distance between the flux lines. The closer together the flux lines are, the larger the magnitude of the field and a further separation between these flux lines simply indicates a decrease in the magnitude of the field. Flux representations of uniform (of the same magnitude) and nonuniform fields are shown in Figures 1.26b and 1.26c, respectively.

It should be emphasized that the reason for our desire to develop such graphical representation is simply to help us visualize the quantitative properties of an existing field. For example, if we reexamine our previous representation of the electric field shown in Figure 1.26d which is due to a point charge Q, we can clearly see that this field is radially directed away from the point charge (as expected from Coulomb's law) and that the magnitude of this field is decreasing (as judged from the increase in the separation distance between the lines) with the increase in the distance away from the charge. This is also true according to Coulomb's law. From Figure 1.26d, it is also clear that the magnitude of the electric field is constant (equal distance between flux lines) at a fixed distance from the point charge.

For a more accurate description of the flux representation of electric fields, let us define a vector quantity **D** known as the *electric flux density*. **D** has the same direction as **E**, the electric field intensity, and its magnitude is $\mathbf{D} = \epsilon_o\,\mathbf{E}$. From Coulomb's law, $\epsilon_o\,\mathbf{E}$ has the dimension of charge/area. Based on Gauss's law, which we will describe in later sections, the number of the flux lines emanating from a charge $+Q$ is equal

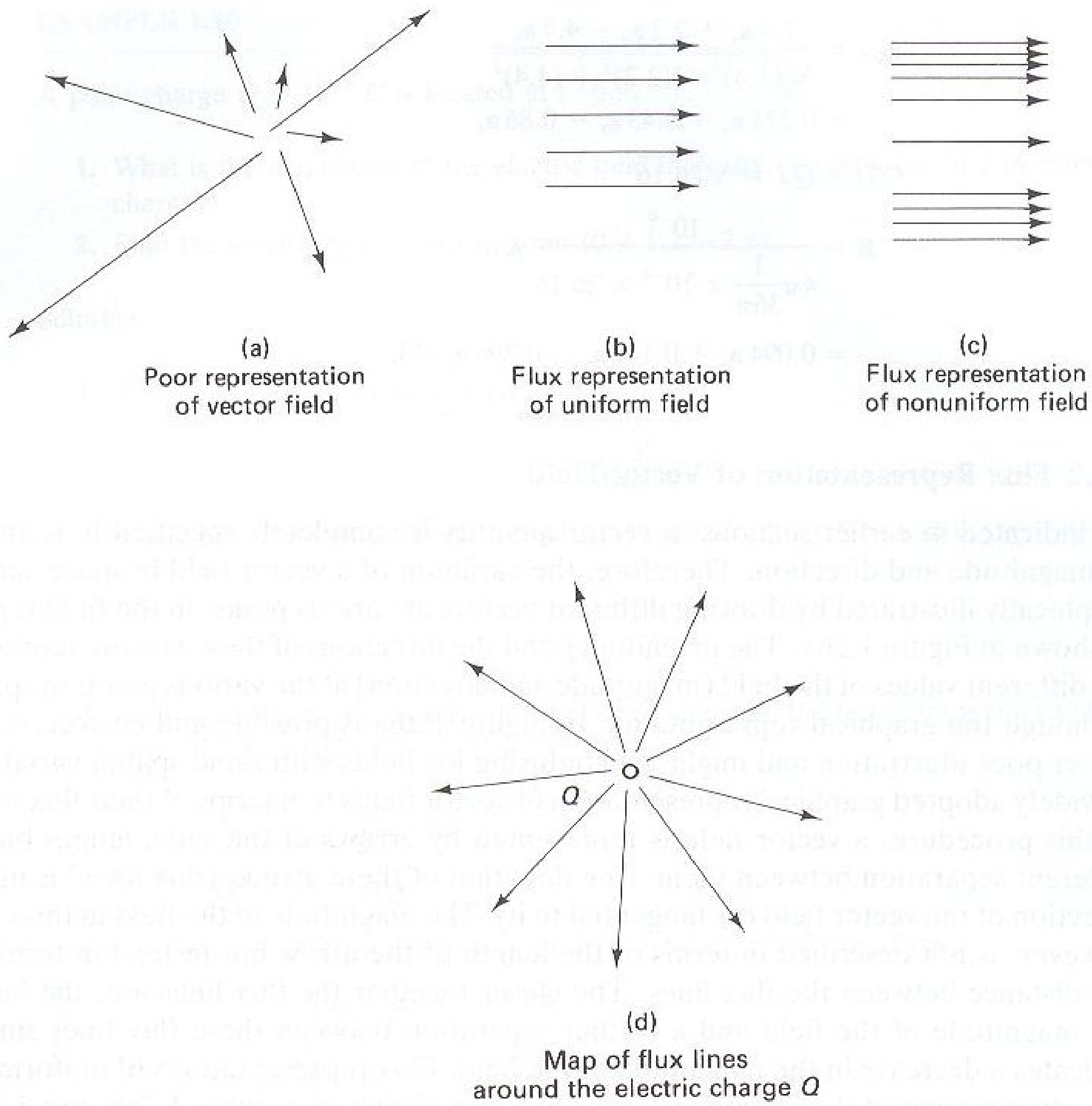

Figure 1.26 Various graphical representations of fields.

to the value of the charge in the SI system of units. Hence, if ϵ is the total number of flux lines

$$\epsilon_{(\text{lines})} = Q(\text{C}) \text{ in the SI system of units}$$

The vector **D** is therefore equal to

$$\mathbf{D} = \frac{\text{charge } Q}{\text{area}} = \frac{\epsilon}{\text{area}} \equiv \text{electric flux density}$$

Hence, **D** is an important parameter in our graphical representation of the field simply because it indicates the number of the flux lines per unit area. This flux representation of a vector field will be further used in future discussions.

EXAMPLE 1.11

Use the flux representation to illustrate graphically the following vector fields:

1. $\mathbf{A} = K\,\mathbf{a}_x$
2. $\mathbf{B} = Kx\,\mathbf{a}_y$
3. $\mathbf{C} = Kx\,\mathbf{a}_x$
4. $\mathbf{D} = \dfrac{K}{\rho}\,\mathbf{a}_\rho$
5. $\mathbf{F} = K\,\mathbf{a}_\rho$

Solution

Before we start graphically representing the given vector fields, let us review the basic rules of the flux representation of a vector field.

1. The direction of the flux lines is in the direction of the vector field or tangential to it.
2. The distance between the flux lines is inversely proportional to the magnitude of the vector field. In other words, the larger the magnitude of the vector field, the smaller the distance between the flux lines will be. With these basic rules in mind, let us now make the desired flux representations.
   (a) $\mathbf{A} = K\,\mathbf{a}_x$ is an x-directed vector with uniform (equal) magnitude everywhere in the Cartesian coordinate system simply because it is independent of the x, y, z variables. A flux representation of the vector **A** is given in Figure 1.27a.
   (b) Vector **B** is in the y direction, and more important is that the magnitude of the vector increases with the increase of x. Vector **B**, therefore, is not uniform and its magnitude increases with the increase in x. The flux lines representing this vector are hence drawn closer as the magnitude of the vector increases with the increase in x. Furthermore, the vector will be directed in the negative y direction for negative values of x. A flux representation of **B** is given in Figure 1.27b.
   (c) In this case, the vector **C** is also directed in the $\mathbf{a}_x$ direction for positive values of x and in the $-\mathbf{a}_x$ direction for negative values of x. Furthermore, the magnitude of the vector increases with the increase in x. This increase in the magnitude of vector **C** is represented graphically in Figure 1.27c by decreasing the distance between the flux lines, or in other words, by increasing the number of flux lines with the increase in x.
   (d) Vector field **D** is best graphically illustrated in the cylindrical coordinate system. It is an $\mathbf{a}_\rho$ directed vector, and its magnitude decreases with the increase in ρ. Figure 1.27d illustrates the flux representation of such a vector where it is clear that just by drawing the $\mathbf{a}_\rho$ directed flux lines, the distance between these lines increases with the increase in ρ, thus demonstrating the decrease in the magnitude of the vector **D** with the increase in ρ.
   (e) The flux representation of the vector **F** is also made in the cylindrical coordinate system because **F** is simply in the $\mathbf{a}_\rho$ direction. To illustrate the uniform magnitude of the vector **F**, however, the distance between the flux lines should be maintained constant. This is achieved graphically by drawing more and more flux

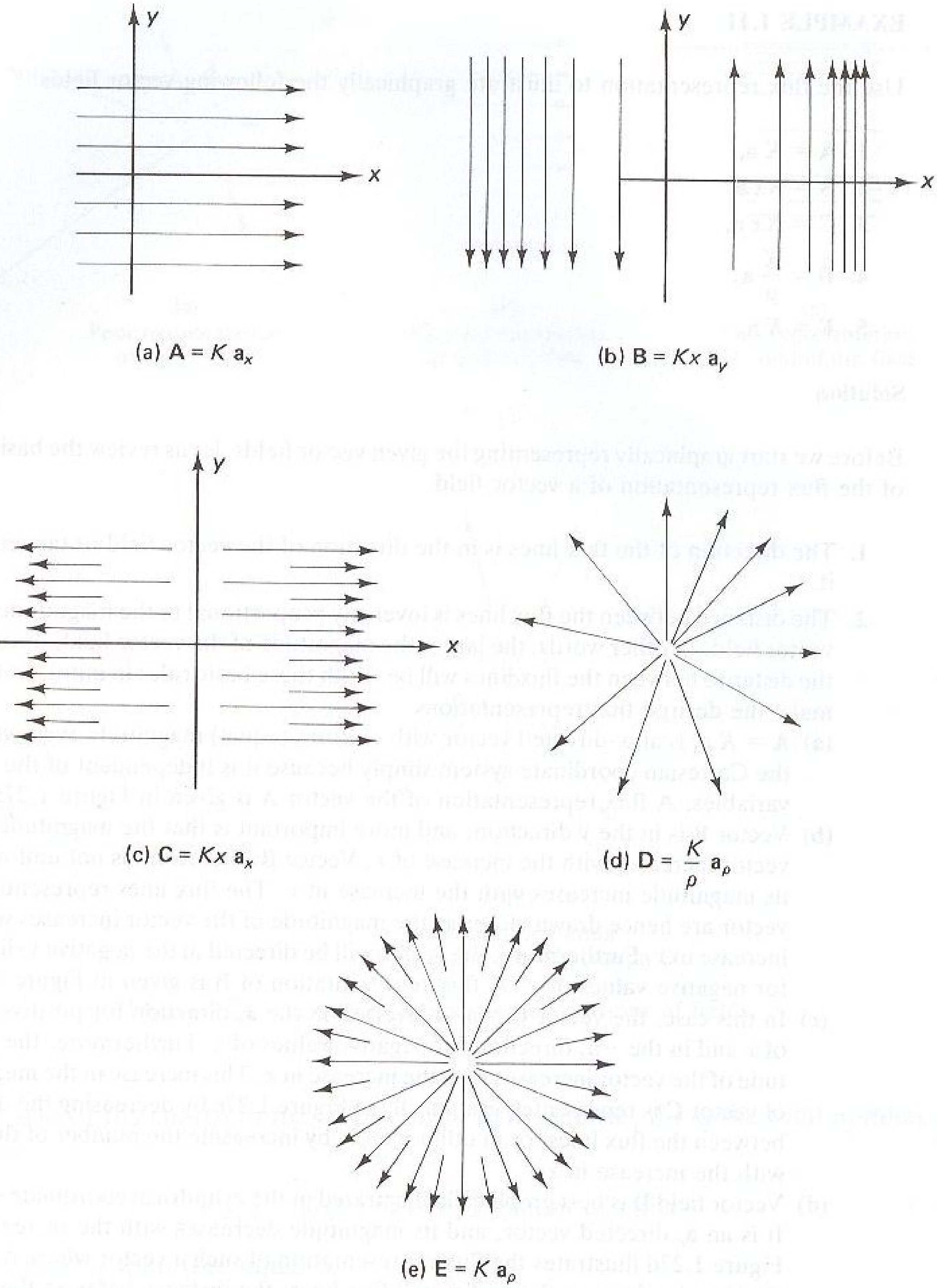

Figure 1.27 Flux representation of various vectors.

lines with the increase in ρ, as shown in Figure 1.27e, so as to maintain the density of the flux lines (i.e., number of flux lines per unit area) almost constant throughout Figure 1.27e.

1.6.3 Magnetic Field

The concept of a field should be familiar by now. Fields really possess no physical basis, because the physical measurements must always be in terms of the forces that result from these fields. As an example of these fields, we discussed in detail the electric field **E** and the forces associated with static electric charges. Another type of force is the *magnetic force* that may be produced by the steady magnetic field of a permanent magnet, an electric field changing with time, or a direct current. We might all be familiar with the magnetic field produced by a permanent magnet that can be recognized through its force of attraction on iron file placed in the neighborhood of the magnet. This phenomenon has been recognized and reported throughout history. It was only in 1820, however, that Oersted discovered that a magnet placed near a current-carrying wire will align itself perpendicular to the wire. This simply means that the steady electric currents exert forces on permanent magnets similar to those exerted by permanent magnets on each other. Ampere then showed that electric currents also exert forces on each other, and that a magnet can be replaced by an equivalent current with the same result. Biot and Savart quantified Ampere's observations, and in the following section we will discuss their findings. Before going to the next section, however, it is worth mentioning that the magnetic field produced by time-varying electric fields is just a mathematical discovery made by Maxwell through his attempt to unify the laws of electromagnetism available at that time. The hypothesis introduced by Maxwell postulating that time-varying electric fields produce magnetic fields will be discussed in detail later in this chapter. In this section we will focus our discussion on the production of magnetic fields by current-carrying conductors. The fundamental law in this study is Biot-Savart's law, which quantifies the magnetic flux density produced by a differential current element.

Biot-Savart's Law. The Biot-Savart law quantifies the magnetic flux density $d\mathbf{B}$ produced by a differential current element $\mathbf{I}d\ell$. The experimental law was introduced to describe the force on a small magnet owing to the magnetic flux produced from a long conductor carrying current **I**. If each of the poles of a small magnet has a strength m, the force **F** caused by the flux **B** is given by

$$\mathbf{F} = m\mathbf{B}$$

This force law is clearly analogous to Coulomb's law for electrostatic field. In this case, the electric force is equal to the charge Q multiplied by the electric field intensity **E**. Hence, $\mathbf{F} = Q\mathbf{E}$.

To quantify the experimental observations by Biot and Savart, the force $d\mathbf{F}$ owing to the magnetic flux $d\mathbf{B}$ produced by a differential current element $\mathbf{I}d\ell$, as shown in Figure 1.28, is found to have the following characteristics.

1. It is proportional to the product of the current, the magnitude of the differential length, and the sine of the angle between the current element and the line connecting the current element to the observation point P.
2. It is inversely proportional to the square of the distance from the current element to the point P.

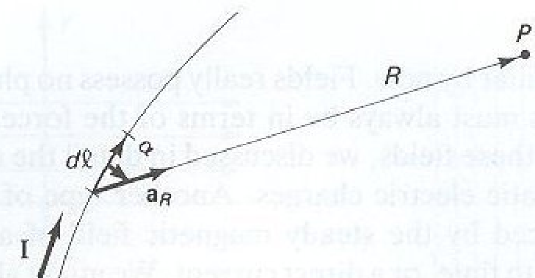

Figure 1.28 The magnetic field intensity at a point P owing to a current element $\mathbf{I}d\ell$. $\mathbf{a}_R$ is a unit vector between the element and the observation point P.

3. The direction of the force is normal to the plane containing the differential current element and a unit vector from the current element to the observation point P. It also follows the right-hand rule from $\mathbf{I}d\ell$ to the line from the filament to P. Hence,

$$|d\mathbf{F}| = |md\mathbf{B}| = \frac{m\mu_o}{4\pi}\frac{I\,d\ell\,\sin\alpha}{r^2}$$

Because the direction of the force $d\mathbf{F}$ is perpendicular to $\mathbf{I}d\ell$ and $\mathbf{a}_R$, a compact expression of the force may take the form

$$\boxed{d\mathbf{F} = md\mathbf{B} = m\mu_o\frac{\mathbf{I}d\ell \times \mathbf{a}_R}{4\pi R^2}}$$

where $\mu_o/4\pi$ is the constant of proportionality.

The following examples will illustrate the use of Biot-Savart's law in calculating magnetic fields from current carrying conductors.

EXAMPLE 1.12

Let us use Biot-Savart's law to find the magnetic flux density produced by a single turn loop carrying a current **I**. We will limit the calculation to the magnetic field along the axis of the loop.

Solution

The magnetic field resulting from the current element 1 ($\mathbf{I}d\ell_1$), which is located at an angle ϕ in Figure 1.29 is given according to Biot-Savart's law by

$$d\mathbf{B}_1 = \frac{\mu_o \mathbf{I}d\ell \times \mathbf{a}_R}{4\pi R^2} = \frac{\mu_o I d\ell \mathbf{a}_\phi \times \mathbf{a}_R}{4\pi(a^2+z^2)}$$

At the element 2, which is symmetrically located with respect to element 1, that is, located at the angle $\phi + \pi$ in Figure 1.29, the magnetic flux density is given by

$$d\mathbf{B}_2 = \frac{\mu_o\, I d\ell\, \mathbf{a}_\phi \times \mathbf{a}_{R_2}}{4\pi(a^2+z^2)}$$

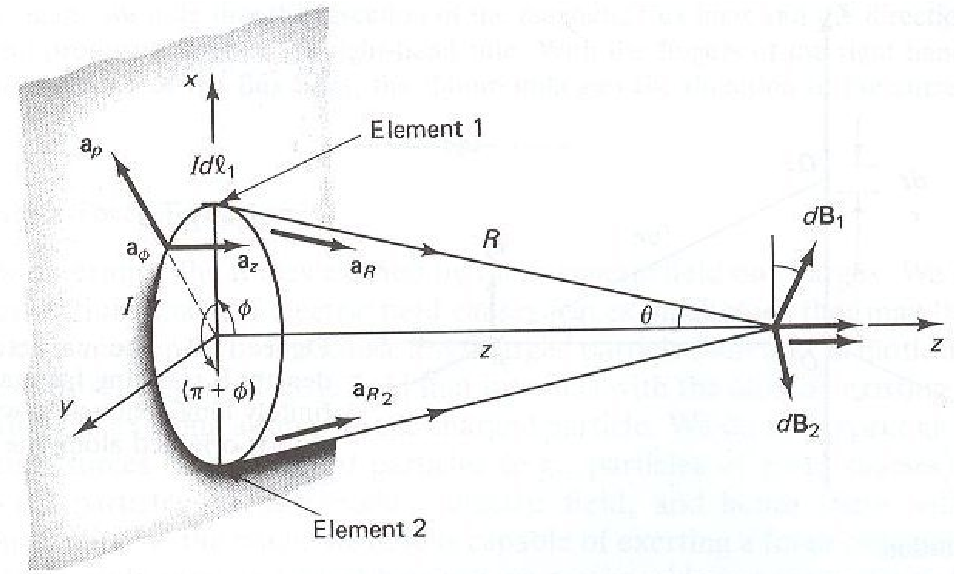

Figure 1.29 Magnetic flux density resulting from a current loop.

From Figure 1.29, it may be seen that the components of $d\mathbf{B}_1$ and $d\mathbf{B}_2$ perpendicular to the z axis cancel and the other components along the z axis, that is, $|d\mathbf{B}_1| \sin\theta$ and $|d\mathbf{B}_2| \sin\theta$, will add, hence,

$$dB_z = \frac{\mu_o I a d\phi \sin\theta}{4\pi(a^2 + z^2)} = \frac{\mu_o I a d\phi}{4\pi(a^2 + z^2)} \frac{a}{(a^2 + z^2)^{1/2}}$$

$$= \frac{\mu_o I a^2 d\phi}{4\pi(a^2 + z^2)^{3/2}}$$

The total magnetic flux density B_z is obtained by integrating dB_z with respect to ϕ from 0 to 2π. Because dB_z is independent of ϕ, we simply multiply dB_z by 2π, hence,

$$B_z = \frac{\mu_o I a^2}{4\pi(a^2 + z^2)^{3/2}} 2\pi = \frac{\mu_o I a^2}{2(a^2 + z^2)^{3/2}}$$

or

$$\mathbf{B} = \frac{\mu_o I a^2}{2(a^2 + z^2)^{3/2}} \mathbf{a}_z$$

This result indicates that the direction of the current flow and the direction of the resulting magnetic field are according to the right-hand rule. When the fingers of the right hand are folded in the direction of the current flow, the thumb will point to the direction of the magnetic flux density **B**.

EXAMPLE 1.13

An infinitely long conducting wire carrying a constant current I and is oriented along the z axis as shown in Figure 1.30. Determine the magnetic flux density at P.

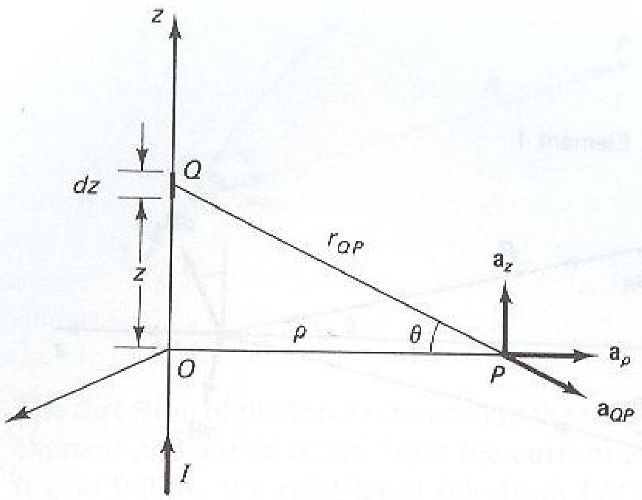

Figure 1.30 The magnetic flux density **B** resulting from an infinitely long conducting wire. The wire is oriented along the positive z axis.

Solution

Because the wire is infinitely long and due to symmetry around the wire, the resulting magnetic field should be independent of z and ϕ. Hence, without loss of generality, we will place P on the $z = 0$ plane. Let us consider an incremental current element Idz located at Q, which is a distance z from the origin O. The unit vector in the direction joining the incremental current element to the field P is

$$\mathbf{a}_{QP} = \mathbf{a}_\rho \cos\theta - \mathbf{a}_z \sin\theta$$

$$= \mathbf{a}_\rho \frac{\rho}{r_{QP}} - \mathbf{a}_z \frac{z}{r_{QP}}$$

where $r_{QP} = \sqrt{z^2 + \rho^2}$. The magnetic field resulting from this current element is given according to Biot-Savart's law by

$$d\mathbf{B} = \frac{\mu_o I dz\, \mathbf{a}_z \times \mathbf{a}_{QP}}{4\pi r_{QP}^2}$$

Substituting $\mathbf{a}_{QP}$ and noting that $\mathbf{a}_z \times \mathbf{a}_\rho = \mathbf{a}_\phi$, and $\mathbf{a}_z \times \mathbf{a}_z = 0$, we obtain

$$d\mathbf{B} = \frac{\mu_o I \rho dz}{4\pi r_{QP}^3} \mathbf{a}_\phi$$

The total magnetic field from the current line is obtained by integrating the contributions from all elements along the line, hence,

$$B_\phi = \frac{\mu_o I \rho}{4\pi} \int_{-\infty}^{\infty} \frac{dz}{(z^2 + \rho^2)^{3/2}}$$

$$= \frac{\mu_o I \rho}{4\pi} \frac{z}{\rho^2 (z^2 + \rho^2)^{1/2}} \Bigg|_{z=-\infty}^{\infty}$$

$$B_\phi = \frac{\mu_o I}{2\pi\rho} \text{ Wb/m}^2$$

or

$$\mathbf{B} = \frac{\mu_o I}{2\pi\rho} \mathbf{a}_\phi$$

Once again, we note that the direction of the magnetic flux lines and the direction of the current producing it obey the right-hand rule. With the fingers of the right hand folded in the direction of the flux lines, the thumb indicates the direction of the current flow.

1.6.4 Lorentz Force Equation

Let us now determine the forces exerted by the magnetic field on charges. We learned in previous sections that the electric field causes forces on charges that may be either stationary or in motion. This is because any charged particle (whether in motion or not) is capable of producing an electric field that interacts with the already existing electric field, resulting in exerting a force on the charged particle. We do not expect an electric field to exert forces on uncharged particles (e.g., particles of given masses) simply because such particles do not produce electric field, and hence there will be no interaction. Similarly, the magnetic field is capable of exerting a force only on moving charges. This result appears logical because we are considering magnetic fields produced by moving charges (currents) and therefore may exert forces on moving charges. The magnetic field cannot be produced from stationary charges and, hence, cannot exert any force on stationary charges.

The force exerted on a charged particle in motion in a magnetic field of flux density **B** is found experimentally to be the following:

1. Proportional to the charge Q, its velocity **v**, the flux density **B**, and to the sine of the angle between the vectors **v** and **B**.
2. The direction of the force is perpendicular to both **v** and **B**, and is given by a unit vector in the direction $\mathbf{v} \times \mathbf{B}$. The force is, therefore, given by

$$\boxed{\mathbf{F} = Q\mathbf{v} \times \mathbf{B}}$$

The force on a moving charge as a result of combined electric and magnetic fields is obtained easily by the superposition of the separate electric and magnetic forces. Hence,

$$\boxed{\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$$

This equation is known as *Lorentz force* equation, and its solution is required in determining the motion of a charged particle in combined electric and magnetic fields.

1.6.5 Differences in Effect of Electric and Magnetic Fields on Charged Particles

The force exerted by the magnetic field is always perpendicular to the direction in which the particle is moving. This force, therefore, does not change the magnitude of the particle's velocity because the work dW done on the particle or the energy delivered to it by the magnetic field is always zero.

$$dW = \mathbf{F} \cdot d\ell = q\mathbf{v} \times \mathbf{B} \cdot \mathbf{v}dt = 0$$

TABLE 1.1 COMPARISON BETWEEN THE ELECTRIC AND MAGNETIC FIELDS

Electric field	Magnetic field
1. Can be produced by charged particles moving or stationary.	Can be produced by direct current that can be attributed to only moving charges.
2. The direction of the force exerted is along the line joining the two charges and is, therefore, independent of the direction of motion of the charged particle.	The force is always perpendicular to the direction of the velocity of the particle.
3. Electric field force causes energy transfer between the field and the charged particle.	The work done on the charged particle is always equal to zero. This is because the magnetic force is always perpendicular to the velocity and hence cannot change the magnitude of the particle velocity.

The magnetic field may, however, deflect the trajectory of the particle's motion but not change the total energy or the total velocity.

The electric field, conversely, exerts a force on the particle that is independent of the direction in which the particle is moving. A velocity component along the direction of the electric field can be generated. The electric field, therefore, causes an energy transfer between the field and the particle. Some fundamental differences between the electric and magnetic fields are summarized in Table 1.1.

To enhance our understanding of the electric and magnetic fields and the nature of their interaction with charged particles further, let us solve the following additional examples.

EXAMPLE 1.14

Consider a particle of mass m and charge q moving in a magnetic field that is oriented in the z direction. The magnetic flux density is given by $\mathbf{B} = B_o \mathbf{a}_z$. If the particle has an initial velocity $\mathbf{v} = v\,\mathbf{a}_x$ (i.e., at $t = 0$), describe the motion of the particle under the influence of the magnetic field.

Solution

From Newton's law and Lorentz force

$$m\mathbf{a} = q\,\mathbf{v} \times \mathbf{B} \tag{1.7}$$

where $\mathbf{a}$ is the particle's acceleration. Expressing equation 1.7 in terms of its components, we obtain

$$m\left(\frac{dv_x}{dt}\mathbf{a}_x + \frac{dv_y}{dt}\mathbf{a}_y + \frac{dv_z}{dt}\mathbf{a}_z\right) = q\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ v_x & v_y & v_z \\ 0 & 0 & B_o \end{vmatrix} \tag{1.8}$$

$$= q(v_y B_o\,\mathbf{a}_x - v_x B_o\,\mathbf{a}_y + 0\,\mathbf{a}_z)$$

Note that although the initial velocity of the particle is in the x direction, we considered all the velocity components in the $\mathbf{v} \times \mathbf{B}$ expression because the velocity of the particle under the influence of the magnetic field is unknown and it is likely that the magnetic field would deflect the particle's trajectory thus generating other velocity components.

Now equating the various components of equation 1.8, we obtain

$$m\frac{dv_x}{dt} = q\,v_y\,B_o \tag{1.9a}$$

$$m\frac{dv_y}{dt} = -q\,v_x\,B_o \tag{1.9b}$$

$$m\frac{dv_z}{dt} = 0 \tag{1.9c}$$

From equation 1.9c it is clear that by integrating with respect to time, we will obtain v_z = constant. Hence, if the particle has an initial velocity in the direction of the magnetic field (z direction), this component of velocity will continue to be constant and unchanged under the influence of the magnetic field. If, conversely, no component of the velocity is initially in the z direction, that is, along the magnetic field, this component will continue to be zero even after the interaction of the charged particle with the magnetic field.

With this in mind regarding the component of the velocity v_z, let us solve equations 1.9a and b for the other two components of the velocity v_x and v_y. Differentiating equation 1.9a once more with respect to t and substituting equation 1.9b for dv_y/dt, we obtain

$$m\frac{d^2 v_x}{dt^2} = -\frac{q^2 B_o^2}{m} v_x$$

or

$$\frac{d^2 v_x}{dt^2} + \frac{q^2 B_o^2}{m^2} v_x = 0 \tag{1.10}$$

A solution of equation 1.10 is in the form

$$v_x = A_1 \cos\omega_o t + A_2 \sin\omega_o t \tag{1.11}$$

where $\omega_o = qB_o/m$ and A_1 and A_2 are two unknown constants to be determined from the initial conditions of the velocity. Substituting v_x in equation 1.9b, we obtain v_y in the form

$$v_y = -\frac{qB_o}{m}\left(+ A_1\frac{\sin\omega_o t}{\omega_o} - A_2\frac{\cos\omega_o t}{\omega_o}\right) \tag{1.12}$$

$$= -A_1 \sin\omega_o t + A_2 \cos\omega_o t$$

To determine A_1 and A_2, let us use the initial conditions of the velocity. At $t = 0$, $\mathbf{v} = v\,\mathbf{a}_x$, and $v_y = 0$, substituting these initial conditions in equations 1.11 and 1.12, we obtain $A_2 = 0$ and $A_1 = v$. The expressions for v_x and v_y are therefore given by

$$v_x = v\cos\omega_o t \qquad \text{and} \qquad v_y = -v\sin\omega_o t$$

The particle's total velocity in the magnetic field is, therefore,

$$\mathbf{v} = v\cos\omega_o t\,\mathbf{a}_x - v\sin\omega_o t\,\mathbf{a}_y \tag{1.13}$$

If we plot the variation of the particle's velocity as a function of time, we can easily see that the particle is rotating in the clockwise direction around the magnetic field as shown in Figure 1.31.

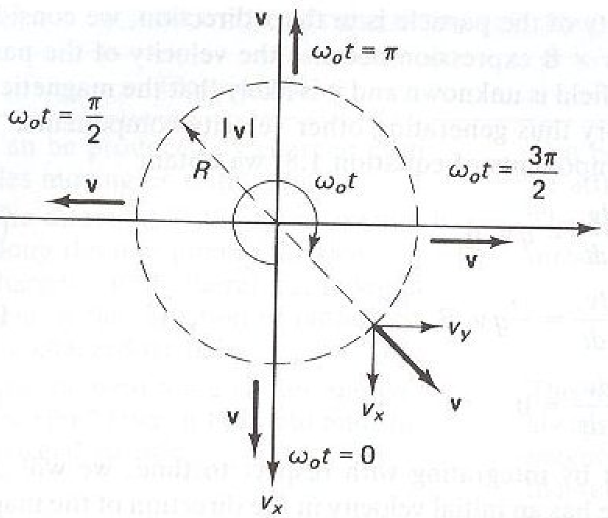

Figure 1.31 Motion of positively charged particle in a constant magnetic field. If the magnetic field is oriented along the positive z axis, the particle moves in a circular path in the clockwise direction.

TABLE 1.2 VELOCITY COMPONENTS AND DIRECTION AS A FUNCTION OF TIME

$\omega_o t$	$\|\mathbf{v}\|$	Direction
0	v	$\mathbf{a}_x$
$\frac{\pi}{2}$	v	$-\mathbf{a}_y$
π	v	$-\mathbf{a}_x$
$\frac{3\pi}{2}$	v	$\mathbf{a}_y$
2π	v	$\mathbf{a}_x$
ϕ	v	Has both $\mathbf{a}_x$ and $\mathbf{a}_y$ components

From Table 1.2 and by noting that

$$|\mathbf{v}| = v\sqrt{\cos^2 \omega_o t + \sin^2 \omega_o t}$$
$$= v$$

it is clear that the magnitude of the particle's velocity is always constant and is equal to the initial velocity. Its components, however, vary as the particle presses around the magnetic field vector in a circular trajectory. The angular velocity ω_o is called the *cyclotron frequency*. The radius of the circle in which the particle travels around the magnetic field is

$$R = \frac{v}{\omega_o}$$

This example simply emphasizes the statement made in the previous section that the magnetic field may deflect the particle's trajectory but not change its velocity—that is, causes no energy transfer from or to the particle.

EXAMPLE 1.15

A charge q of mass m is injected into a field region containing perpendicular electric and magnetic fields. When the charge velocity at any point along the motion path is $\mathbf{v} = v_x\,\mathbf{a}_x$, the observed acceleration is $\mathbf{a} = a_x\,\mathbf{a}_x + a_y\,\mathbf{a}_y$.

Find an **E** and **B** combination that would generate this acceleration **a**.

Solution

When the velocity has only one component in the x direction, the acceleration was found to have two components.

$$\mathbf{a} = a_x\,\mathbf{a}_x + a_y\,\mathbf{a}_y$$

In the presence of both **E** and **B** fields, the force is given by

$$\mathbf{F} = m(a_x\,\mathbf{a}_x + a_y\,\mathbf{a}_y) = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Because **v** has only one component in the x direction, then the magnetic field force cannot be responsible for the x component of the force. The electric field force is therefore the cause of the x component of the acceleration. Hence,

$$\mathbf{E} = \frac{m\,a_x}{q}\,\mathbf{a}_x$$

and

$$\mathbf{B} = -\frac{m\,a_y}{q\,v_x}\,\mathbf{a}_z$$

To explain further the reason for **B** to have only an $\mathbf{a}_z$ component, let us assume that **B** has B_y and B_z components. We should note that **B** has no B_x component because **E** (has only x component) and **B** are perpendicular to each other. Now if we assume that **B** has B_y and B_z components, from $\mathbf{v} \times \mathbf{B}$ determinant, there should be an $\mathbf{a}_z$ component of force or consequently an $\mathbf{a}_z$ component of acceleration.

$$\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ v_x & 0 & 0 \\ 0 & B_y & B_z \end{vmatrix} = -v_x\,B_z\,\mathbf{a}_y + v_x\,B_y\,\mathbf{a}_z$$

Because the $\mathbf{a}_z$ component of the acceleration is zero, B_y has therefore to be zero.

EXAMPLE 1.16

Two small balls of masses m have a charge Q each, and are suspended at a common point by thin filaments, each of length ℓ. Assuming that the charges are to be located approximately at the centers of the balls, find the angle α between the filaments. (Assume α to be small.) *Note*: Such a system can be used as a primitive device for measuring charges and potentials and is called an electroscope.

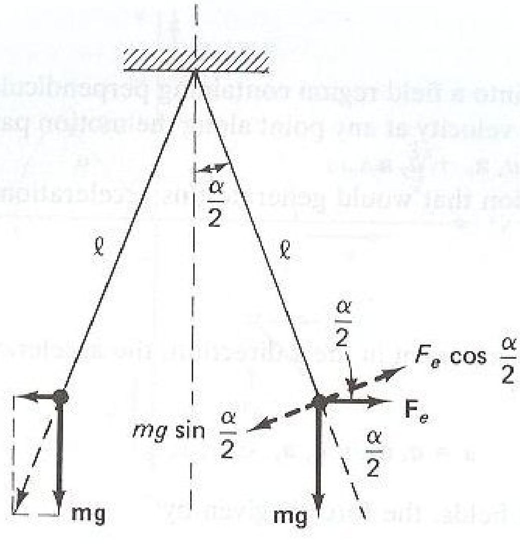

Figure 1.32 The electroscope.

Solution

Because the two balls are charged with similar charges, the repulsion force will cause them to separate away from each other. The two balls will reach the equilibrium position when the force perpendicular to each string becomes zero as shown in Figure 1.32. This is simply because this force is responsible for swinging the balls.

From Figure 1.32, it may be seen that the equilibrium position will occur when

$$mg \sin \alpha/2 = F_e \cos \alpha/2$$

The electric force between the two charged balls F_e is given by

$$F_e = \frac{Q^2}{4\pi \epsilon_o (2\ell \sin \alpha/2)^2}$$

Hence, the equilibrium equation reduces to

$$\frac{Q^2}{16\pi \epsilon_o \ell^2 mg} = \frac{\sin^3 \alpha/2}{\cos \alpha/2}$$

For small α, $\cos \alpha/2 = 1$ and $\sin \alpha/2 = \alpha/2$. Thus,

$$\alpha^3 = \frac{Q^2}{2\pi \epsilon_o \ell^2 mg}$$

In the previous sections we familiarized ourselves with the simple rules of vector algebra and the basic concepts of fields. In the following section we will continue our efforts to pave the way for the introduction of Maxwell's equations. Specifically, we will introduce the vector integration as a prerequisite to the discussion of Maxwell's equations in integral form.

1.7 VECTOR INTEGRATION

Besides the vector representation of the electromagnetic field quantities and the ability to transform such representation from one coordinate system to another, it is important that we develop a thorough understanding of basic vector integral and differential operations. Vector differential operations will be discussed in chapter 2 just before the introduction of Maxwell's equations in differential form. In preparation of the introduction of Maxwell's equation in integral form, we introduce vector integral operations next.

1.7.1 Line Integrals

The scalar line integral $\int_a^b A(\ell)\,d\ell$ (where ℓ is the length of the contour and a and b are the two end points along the path of integration) is defined as the limit of the sum $\sum_{i=1}^{N} A(\ell_i)\,\Delta\ell_i$ as $\Delta\ell_i \to 0$. $A(\ell_i)$ is the value of $A(\ell)$ evaluated at the point ℓ_i within the segment $\Delta\ell_i$. This simply means that in evaluating $\int_c A(\ell)\,d\ell$, we divide the contour of integration c into N segments, as shown in Figure 1.33, evaluate the scalar quantity $A(\ell_i)$ at the center of each element, multiply $A(\ell_i)$ by the length of the element $\Delta\ell_i$, and add the contributions from all the segments. The sum of these contributions will equal exactly the line integral of the scalar quantity in the limit when the lengths of these elements $\Delta\ell_i$ approach zero. Hence,

$$\int_c A(\ell)\,d\ell = \lim_{\substack{\Delta\ell_i \to 0 \\ N \to \infty}} \sum_{i=1}^{N} A(\ell_i)\,\Delta\ell_i$$

A simple example of this line integral is the evaluation of $\int_c d\ell$ where the contour c is given by the curve shown in Figure 1.34. The element of length $d\ell$ in this case is given by $\rho\,d\phi$ where $\rho = 1$ along the given contour c. Therefore,

$$\int_c d\ell = \int_0^{\pi/2} \rho|_{\rho=1}\,d\phi = \int_0^{\pi/2} d\phi = \frac{\pi}{2}$$

If we follow the physical reasoning behind the evaluation of the line integral of a scalar quantity as described earlier, it can be shown that the line integral of the form $\int_c d\ell$ is simply the length of the contour c. Hence, if c is given by the curve shown in Figure 1.34, then

$$\int_c d\ell = \frac{\text{Circumference of circle}}{4}$$

$$= \frac{2\pi(1)}{4} = \frac{\pi}{2}$$

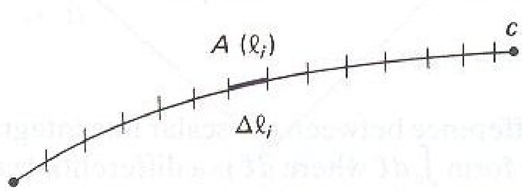

Figure 1.33 An approximate procedure for calculating a scalar line integral.