EE 434 Electromagnetic waves, Spring 2009
Handout 2, January 28, 2009

1 Numerical discretization of Poisson's and Laplace's equations

We can solve electrostatic problems in terms of the potential by starting with Gauss' law and substituting in the expression for the electric field in terms of the potential,

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot (-\epsilon \nabla \Phi) = \rho$$

If we assume a constant value of ϵ, we can simplify to

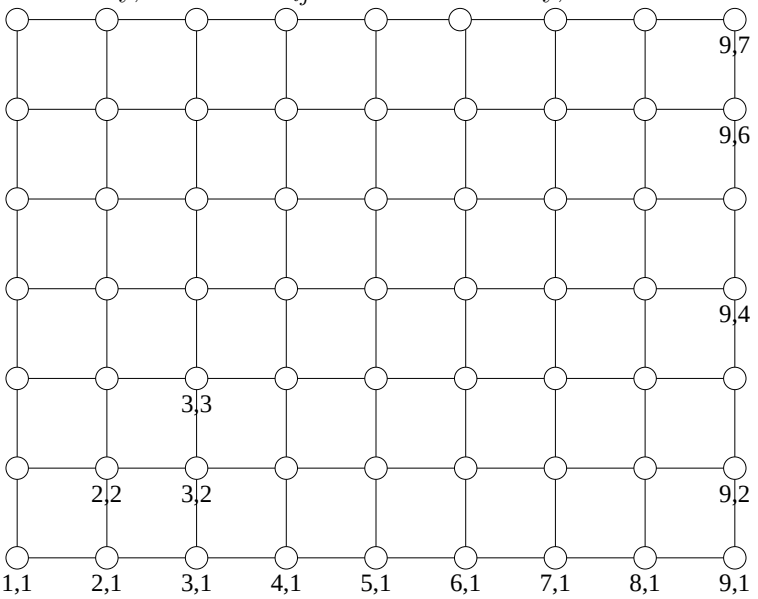
$$\nabla^2 \Phi = -\frac{\rho}{\epsilon}$$

This is Poisson's equation. The special case of $\rho = 0$ is called Laplace's equation, and we solve Laplace's equation to get the potential in the space between conductors, where there are no free charges.

In order to solve Laplace's equation (or Poisson's equation) numerically, we need to convert it to a discrete form which can be processed by a computer. For simplicity, let's consider a two-dimensional problem (A 3-dimensional problem is analogous, just a little bit more involved). In that case we can write, in a cartesian coordinate system

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

We will look for a solution a two-dimensional regular cartesian grid, such that instead of $\Phi(x, y)$ defined continuously, we have Φ_{ij} defined discretely, like this



Where Φ is defined at the nodes. In this case it is a 9×7 grid, with 9 nodes along the first (x) axis, and 7 nodes along the second (y) axis. Now we can write the derivatives in terms of

differences. Assume the grid spacing is h, we could approximate the derivative at node $3,3$ along the x-dimension to be

$$\left(\frac{\partial \Phi}{\partial x}\right)_{3,3} = \frac{\Phi_{4,3} - \Phi_{2,3}}{2h}$$

where h is the grid spacing. This is a symmetric difference in which we use the values on either side. We can also compute a derivative just to the left and just to the right of the point $(3,3)$, like this

$$\begin{aligned}\left(\frac{\partial \Phi}{\partial x}\right)_{3,3\,L} =& \frac{\Phi_{3,3} - \Phi_{2,3}}{h} \\ \left(\frac{\partial \Phi}{\partial x}\right)_{3,3\,R} =& \frac{\Phi_{4,3} - \Phi_{3,3}}{h}\end{aligned}$$

Althought these are asymmetric they are useful in compute the second derivative at the point $(3,3)$, like this,

$$\begin{aligned}\left(\frac{\partial^2 \Phi}{\partial x^2}\right)_{3,3} =& \frac{1}{h}\left[\left(\frac{\partial \Phi}{\partial x}\right)_{3,3\,R} - \left(\frac{\partial \Phi}{\partial x}\right)_{3,3\,L}\right] \\ =& \frac{1}{h^2}\left[\Phi_{4,3} + \Phi_{2,3} - 2\Phi_{3,3}\right]\end{aligned}$$

In the general case we get for the second derivative along the x-axis

$$\left(\frac{\partial^2 \Phi}{\partial x^2}\right)_{i,j} = \frac{1}{h^2}\left[\Phi_{i+1,j} + \Phi_{i-1,j} - 2\Phi_{i,j}\right]$$

and along the y-axis

$$\left(\frac{\partial^2 \Phi}{\partial y^2}\right)_{i,j} = \frac{1}{h^2}\left[\Phi_{i,j+1} + \Phi_{i,j-1} - 2\Phi_{i,j}\right]$$

Now we can write down the two-dimensional Laplace (or Poisson) equation in dicrete form

$$\nabla^2 \Phi = \frac{1}{h^2}\left[\Phi_{i+1,j} + \Phi_{i-1,j} + \Phi_{i,j+1} + \Phi_{i,j-1} - 4\Phi_{i,j}\right] = 0$$

Now, solving the Laplace equation means finding $\Phi_{i,j}$ for all (i,j) points such that

1. The potential, Φ is equal to the prescribed value at boundaries (for example where we define conductors and their potentials).

2. The potential satisifie the following relation for all (i,j)

$$\Phi_{i,j} = \frac{1}{4}\left[\Phi_{i+1,j} + \Phi_{i-1,j} + \Phi_{i,j-1} + \Phi_{i,j+1}\right]$$

2 Solution of Laplace's equation with relaxation

The question that remains is how we solve this problem: the potential is pre-defined at some nodes and for the rest of the nodes it must be true that the potential is the average of the potential of it's four neighbors.

There are many ways of solving this problem, ranging from matrix calculations (as described in the textbook), to sophisticated numerical techniques. Here I will describe a very simple approach, which is not particularly fast or elegant, but which is simple to implement and works.

Create two arrays with the same dimensions. One array, call it P, will contain the values of the potential, and the other array, call it M, will contain a marker, 0 or 1. Initialize both arrays to have zero values all over. Next decide what your boundary conditions are. For example, you could decide that the edge of your simulation must have zero potential, and that various nodes or collections of nodes also have fixed potential. Write those potentials into the array P, and place a 1 in the same array elements in M. Now we are ready to begin iterating. You will repeat the following procedure many times until either (1) the maximum change in the potential from one iteration to the next is very small, or (2) you have reched a specified maximum number of iterations.

The procedures is as follows. Loop over all the elements of the array. If $M(i, j) = 0$, then change the value of the potential to be the average of the four neighbors and make note of how much you changed it. Keep track of the largest change as you will use that to decide when to stop the iterations.

That is it. Once you have iterated enough that the change is small you are done and you (hopefully) have a good solution to the potential under the stated boundary conditions.

Note, that since we are averaging over the four neighbors, this will really only work if the edge of the array is a boundary condition, otherwise we will have a situation where we will be addressing array elements beyond the edge of the array.