EE 434 Electromagnetic Waves, Spring 2009
Exam #1, 2009/2/20
Solutions

Question 1: Electrostatics

Consider two perfectly conducting spheres of diameter a, separated (center to center) by a distance d. One sphere has a charge Q, and the other has a charge $-Q$. Assume that the spheres are far enough apart and small enough that the charge is uniformly distributed on the surfaces of the spheres.

(a) What is the potential on the surface of each sphere, and the potential difference between the spheres? Remember to include the effect of the other sphere as its potential at the average distance between sphere centers.

The potential from a spherical charge is

$$\Phi(r) = \frac{Q}{4\pi\epsilon_0 r}$$

At the positively charged sphere the total potential is

$$V_+ = \frac{Q}{4\pi\epsilon_0 a} - \frac{Q}{4\pi\epsilon_0 d}$$

At the negatively charged sphere the total potential is

$$V_- = -\frac{Q}{4\pi\epsilon_0 a} + \frac{Q}{4\pi\epsilon_0 d}$$

The potential difference between the spheres is

$$V = V_+ - V_- = \frac{2Q}{4\pi\epsilon_0 a} - \frac{2Q}{4\pi\epsilon_0 d}$$

(b) What is the capacitance of the system?

The capacitance of the system is

$$C = \frac{Q}{V} = \frac{1}{\frac{2}{4\pi\epsilon_0 a} - \frac{2}{4\pi\epsilon_0 d}}$$

(c) If we assume $Q = 1\,\mu\mathrm{C}$, $d = 10\,\mathrm{m}$, and $a = 0.1\,\mathrm{m}$, how much energy did it take to charge the spheres?

The amount of energy required is

$$W = \frac{1}{2}QV = \frac{Q^2}{4\pi\epsilon_0 a} - \frac{Q^2}{4\pi\epsilon_0 d}$$

Inserting known values we get

$$W = \frac{(1\times 10^{-6})^2}{4\pi \times 8.854 \times 10^{-12}}\left(\frac{1}{0.1} - \frac{1}{10}\right) = 0.09\,\mathrm{J}$$

(d) Carefully explain why you cannot in general assume that the surface charge density on the spheres is uniform. Hint: Argue your point in terms of the relationship between electric field and potential and in terms of the relationship between electric field and surface charge density.

The problem is that on the surfaces of the spheres facing each other the perpendicular electric field will be larger than on the surfaces facing away from each other. This in turn causes the charges to move to prevent electric fields from existing inside the spheres.

Question 2: Magnetostatics

Three currents loops. All three loops have their center at the origin of the coordinate system. Loop 1 is in the XY plane, has diameter $a_1 = 1\,\mathrm{m}$, and carries current $I_1 = 1\,\mathrm{A}$. Loop 2 is in the YZ plane, has diameter $a_2 = 0.5\,\mathrm{m}$, and carries current $I_2 = 2\,\mathrm{A}$. Loop 3 is in the XZ plane, has diameter $a_3 = 0.25\,\mathrm{m}$, and carries the current $I_3 = 4\,\mathrm{A}$.

(a) What is $\vec{A}$ at the origin of the coordinate system?

The magnetic potential is computed as

$$\vec{A}\left(\vec{r}\right) = \int_L \frac{I\left(\vec{r}'\right)\, d\vec{l}'}{\left|\vec{r} - \vec{r}'\right|}$$

Notice that each current element on a circle is at the same distance from the origin, and that current elements located 180° from each other cancel out. The conclusion is that each circle contributes zero to the magnetic potential at the center of the circle, so

$$\vec{A}\left(\vec{0}\right) = \vec{0}$$

(b) What is $\vec{B}$ at the origin of the coordinate system?

The magnetic field is computed with the Biot-Savart law,

$$\vec{B}\left(\vec{r}\right) = \frac{\mu_0}{4\pi} \int_L \frac{I\left(\vec{r}'\right)\, d\vec{l}' \times \hat{R}}{\left|\vec{r} - \vec{r}'\right|^2}$$

We note that due to symmetry each current loop creates a field at the center which is perpendicular to the loop. Since I flows in the right-hand direction around the axis, the field is in the positive direction around the axis. Also, $d\vec{l}' \times \hat{R}$ can be replaced with $d\vec{l}\hat{a}$, where $\hat{a}$ is the appropriate axis unit vector. Finally, $\left|\vec{r} - \vec{r}'\right| = a$. We can now write in the general case

$$B_a = \frac{\mu_0}{4\pi} \int_L \frac{I\, dl}{a^2} = \frac{\mu_0 I}{4\pi a^2} \int_0^{2\pi} a\, d\phi = \frac{\mu_0 I}{4\pi a^2} 2\pi a = \frac{\mu_0 I}{2a}$$

Now for the XY loop, loop 1, we get

$$B_z = \frac{\mu_0 I_1}{2a_1} = \frac{4\pi \times 10^{-7}\, 1}{2 \times 1} = 6.28 \times 10^{-7}\,\mathrm{T} = 62.8\,\mathrm{nT}$$

For the YZ loop, loop 2, we get

$$B_x = \frac{\mu_0 I_2}{2a_2} = \frac{4\pi \times 10^{-7}\, 2}{2 \times 0.5} = 5.03 \times 10^{-6}\,\mathrm{T} = 503\,\mathrm{nT}$$

For the XZ loop, loop 3, we get

$$B_y = \frac{\mu_0 I_3}{2a_3} = \frac{4\pi \times 10^{-7}\, 4}{2 \times 0.25} = 2.0106 \times 10^{-5}\,\mathrm{T} = 20106\,\mathrm{nT}$$

The magnetic field at the origin is thus

$$\vec{B}\left(\vec{0}\right) = (62.8, 503, 20106)\ \mathrm{nT}$$

(c) Use a symmetry argument to determine the mutual inductances M_{12}, M_{13}, and M_{23}.

All of the mutual inductances are zero. The reason is that the magnetic field from each coil must, because of symmetry, be in a cylindrical radial plane everywhere in space. Since every coil is in a cylindrical radial plane relative to every other coil, the amount of magnetic flux through them from the other coils is zero, and thus the mutual inductance is zero.