EE 434 Electromagnetic Waves, Spring 2009
Exam #3, 2009/4/17 solutions

Rules

You may make use of a single sheet of notes/formulas that you prepared ahead of the exam, as well as your calculator. Each question in this exam counts equally toward your grade.

Questions

(1) Sketch the electric field of the TE$_{12}$ mode in a rectangular waveguide.

For the TE_{12} mode the axial electric field is zero and the axial magnetic field amplitude is

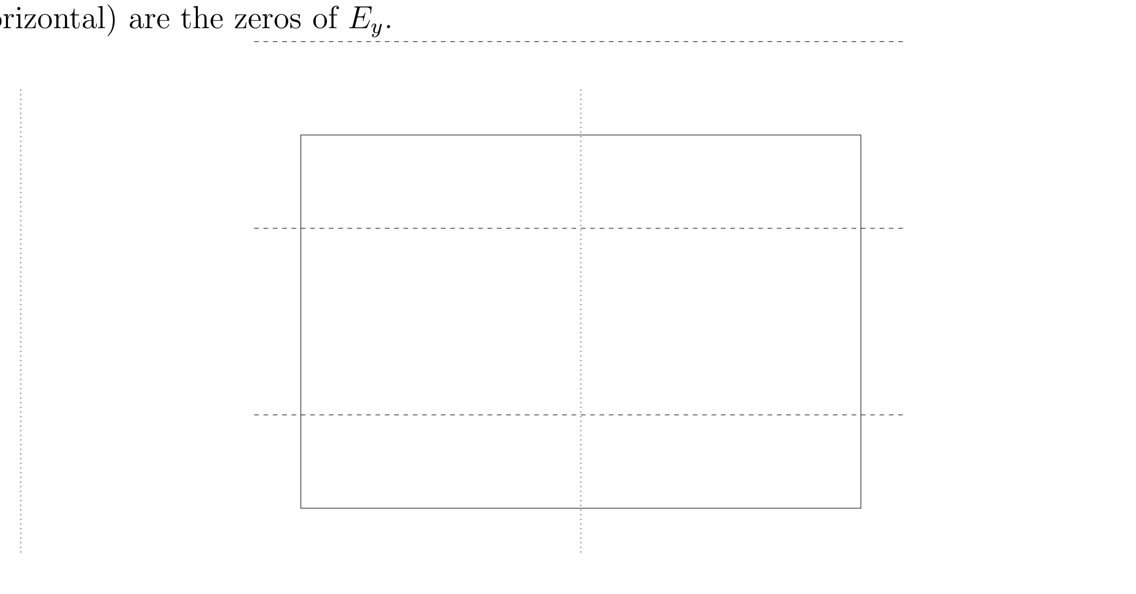
$$H_z(x, y) = A \cos\left(\frac{\pi x}{a}\right) \cos\left(\frac{2\pi y}{b}\right)$$

Now we remember that E_x is the y-derivative of H_z, and E_y is the x-derivative of H_z. There are some signs as well, but we will not worry about those, simply writing

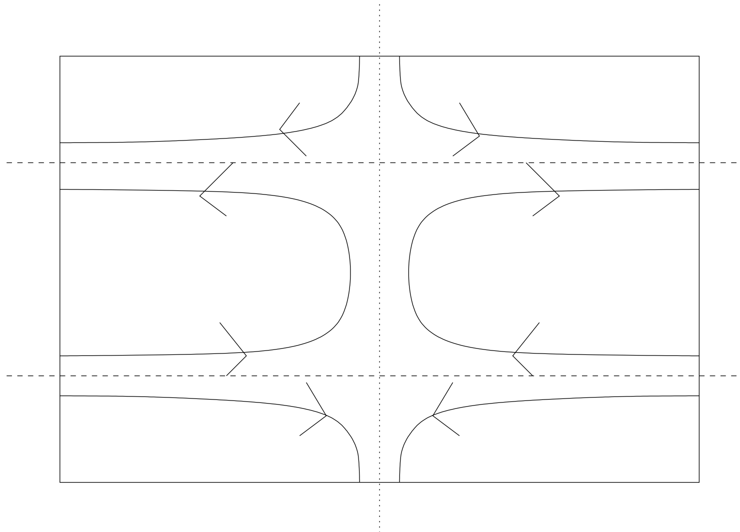
$$E_x(x, y) = A' \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi y}{b}\right)$$

$$E_y(x, y) = A'' \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{2\pi y}{b}\right)$$

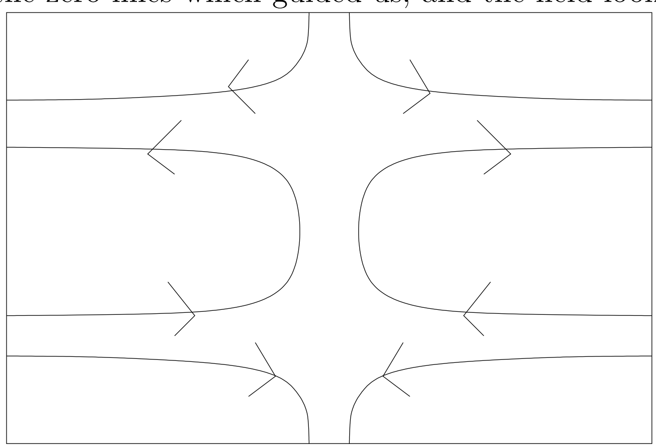
An easy way to begin with this is to plot the lines where E_x and E_y are zero, respectively. In the following figure the dotted lines (vertical) are the zeros of E_x, and the dashed lines (horizontal) are the zeros of E_y.



Note that I drew the zero lines beyond the edge of the waveguide. Now we know from our discussions in class that the field forms cells between the zero lines and that the fields are parallel to the zero lines near the zero lines. Drawing the zero lines beyond the edge of the waveguide makes it more clear that the field lines should intersect the edge of the waveguide perpendicularly. Now we can easily sketch the fields (I will sketch just a single field line in each cell),



Next we can remove the zero lines which guided us, and the field looks like



(2) A TE_{12} mode is propagating in a air-filled rectangular waveguide with $a =$ 0.1 m, and $b = 0.01$ m. If the amplitude of E_x is 1 V/m, what is the amplitude of E_y?

The axial magnetic field of this mode is

$$H_z = A \cos\left(\frac{\pi}{a}x\right) \cos\left(\frac{2\pi}{b}y\right)$$

E_x is the y-derivative of this, and E_y is the x-derivative. Thus, we have

$$E_x = A'\frac{2\pi}{b} \cos\left(\frac{\pi}{a}x\right) \sin\left(\frac{2\pi}{b}y\right) = E_{x0} \cos\left(\frac{\pi}{a}x\right) \sin\left(\frac{2\pi}{b}y\right)$$

and

$$E_y = A'\frac{\pi}{a} \sin\left(\frac{\pi}{a}x\right) \cos\left(\frac{2\pi}{b}y\right) = E_{y0} \sin\left(\frac{\pi}{a}x\right) \cos\left(\frac{2\pi}{b}y\right)$$

where A' is the same constant in each case. From this we see that

$$\frac{E_{x0}}{E_{y0}} = \frac{2a}{b}$$

and thus

$$E_{y0} = E_{x0}\frac{b}{2a} = 1 \times \frac{0.01}{0.2} = 0.05\,\text{V/m}$$

(3) A air-filled rectangular waveguide has dimensions $a = 0.1\,\text{m}$, and $b = 0.05\,\text{m}$. If it is excited with a frequency $\omega = 25 \times 10^9\,\text{s}^{-1}$, what modes are possible?

The possible modes are the ones whose critical frequency is smaller than the excitation frequency.

$$\omega_c = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

Make a table

n / m	0	1	2	3
0	0	9.4	18.9	28.3
1	18.9	21.1	26.6	34.0
2	37.7	38.8	42.1	47.1

Thus, possible modes are TE_{10}, TE_{20}, TE_{01}, TE_{11}, TM_{11}.

(4) What are the phase and group velocities of a $\omega = 12 \times 10^9\,\text{s}^{-1}$ TE_{01} wave in a air-filled rectangular waveguide with $a = 0.1\,\text{m}$ and $b = 0.1\,\text{m}$?

We start with the dispersion relation,

$$\beta = \omega\sqrt{\mu\epsilon}\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}$$

The phase velocity is

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon}\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}}$$

The critical frequency is

$$\begin{aligned} \omega_c =& \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \\ =& 9.4 \times 10^9\,\text{s}^{-1} \end{aligned}$$

so the phase velocity is

$$v_p = \frac{1}{\sqrt{\mu\epsilon}\sqrt{1 - \left(\frac{9.4}{12}\right)^2}} = 4.82 \times 10^8\,\text{m}/texts = 1.61\,\text{c}$$

Next we derive the expression for the group velocity. It is

$$\begin{aligned} v_g =& \frac{d\omega}{d\beta} = \left(\frac{d\beta}{d\omega}\right) = \frac{1}{\sqrt{\mu\epsilon}}\left[\frac{d}{d\omega}\sqrt{\omega^2 - \omega_c^2}\right]^{-1} \\ =& \frac{1}{\sqrt{\mu\epsilon}}\left[\frac{2\omega}{2\sqrt{\omega^2 - \omega_c^2}}\right]^{-1} = \frac{1}{\sqrt{\mu\epsilon}}\frac{\sqrt{\omega^2 - \omega_c^2}}{\omega} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} \end{aligned}$$

Inserting given and computed values we get

$$v_g = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{1 - \left(\frac{9.4}{12}\right)^2} = 1.86 \times 10^8\,\text{m/s} = 0.62\,c$$

(5) You use the TM_{11} mode as a carrier to transmit information through a 100 m long waveguide. If the air-filled rectangular waveguide has dimensions $a = 10$ cm, and $b = 8$ cm, and the wavelength of the mode is 50 cm how long does it take the information to propagate the length of the waveguide?

We compute the critical frequency, use that together with the wavelength to find the frequency, and then use that together with the formula from the previous question to compute the group velocity, which is the propagation velocity of a signal, and then we compute how long it takes to travel the length of the waveguide at that velocity. The wavelength is written as

$$\lambda = \frac{2\pi}{\omega\sqrt{\mu\epsilon}\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)}} = \frac{2\pi}{\sqrt{\mu\epsilon}\sqrt{\omega^2 - \omega_c^2}}$$

Isolate ω,

$$\omega^2 = \left(\frac{2\pi}{\lambda\sqrt{\mu\epsilon}}\right)^2 + \omega_c^2$$

The critical frequency is

$$\omega_c = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2} = 15.077 \times 10^9\,\text{s}^{-1}$$

and the wave frequency is

$$\omega = \sqrt{\left(\frac{2\pi}{0.5\sqrt{\mu\epsilon}}\right)^2 + \omega_c^2} = 15.540 \times 10^9\,\text{s}$$

The group velocity is now

$$v_g = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{1 - \left(\frac{15.077}{15.540}\right)^2} = 7.27 \times 10^7\,\text{ms} = 0.242\,c$$

The travel time for the signal is then

$$\Delta t = \frac{d}{v_g} = \frac{100}{7.27 \times 10^7} = 1.38\,\mu\text{s}$$