EE 434 Electromagnetic Waves, Spring 2009
Practice exam, 2009/5/4

(1) What is a broadside antenna array? What is a end-fire antenna array?
A broadside array is an array whose gain maximum is in a direction perpendicular to the array. A end-fire array is an array whose gain maximum is in a direction parallel to the array.

(2) An antenna (or antenna array) has the directional gain

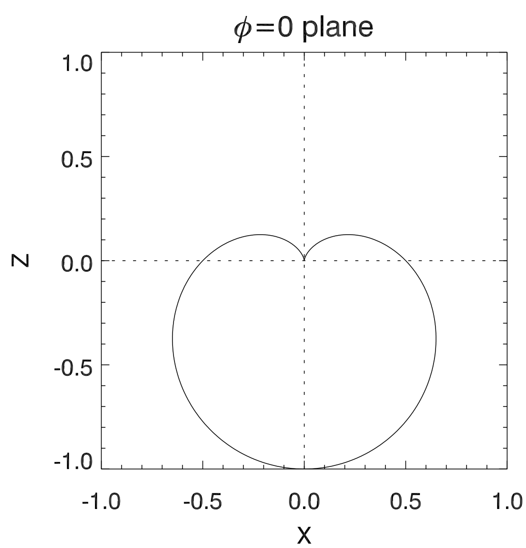
$$\boldsymbol{D(\theta,\phi) = \sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\phi}{2}\right)}$$

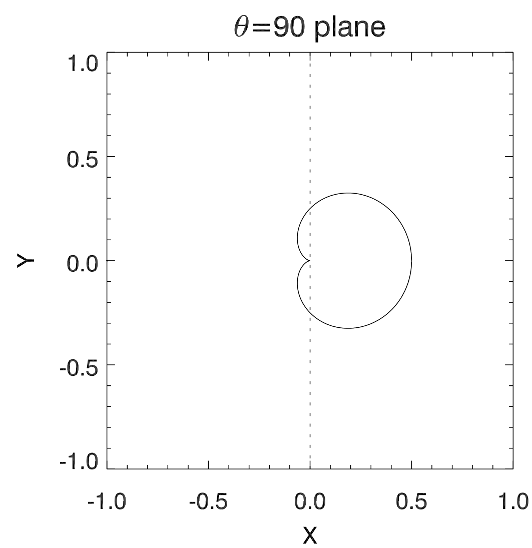
Plot the gain in the $\boldsymbol{\phi = 0}$ and $\boldsymbol{\theta = 90^\circ}$ planes respectively.
In the $\phi = 0$ plane the function looks like

$$D(\theta,0) = \sin^2\left(\frac{\theta}{2}\right)$$

In the $\theta = 90^\circ$ plane the function looks like

$$D(90^\circ,\phi) = \sin\left(45^\circ\right)^2\cos^2\left(\frac{\phi}{2}\right) = \frac{1}{2}\cos^2\left(\frac{\phi}{2}\right)$$





(3) For the same direction gain as in question 2, write an expression for a vector wave electric field which is consistent with this directional gain.
This could for example be an electric field which is oriented along the $\hat{\theta}$ direction, and whose amplitude is K/r times the square root of the directional gain. Then there is also the wave term $\exp\left(j\omega t - j\beta r\right)$. Overall it becomes

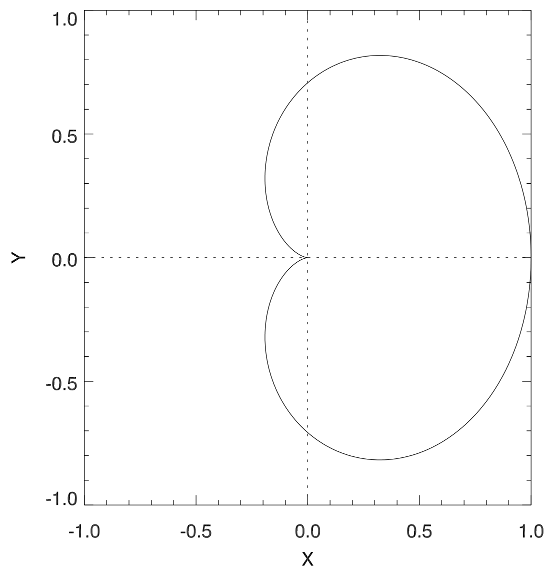
$$\vec{E}\left(r,\theta,\phi\right) = \hat{\theta}\frac{K}{r}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\phi}{2}\right)$$

(4) For an array of two antennas, what spacing, $\boldsymbol{d}$, and phase difference, $\boldsymbol{\psi}$ will result in an end-fire array with a power maximum in the positive direction and minimum in the negative direction.

The array factor for a linear array is

$$\mathrm{AF} = \frac{\sin\left(\frac{N}{2}\left[\beta d\cos\phi - \psi\right]\right)}{\sin\left(\frac{1}{2}\left[\beta d\cos\phi - \psi\right]\right)}$$

where $\beta = \frac{2\pi}{\lambda}$. For $N = 2$ we are thus looking for values of d and ψ that produce a maximum at $\phi = 0$ and a minimum at $\phi = \pi$. A little experimentation leads to $\beta d = \frac{\pi}{2}$ and $\psi = \frac{\pi}{2}$. To verify we plot



(5) Sketch the magnetic field of a TM_{21} mode in a rectangular waveguide. And then the electric field of a TE_{21} mode.

For a TM mode the axial electric field is

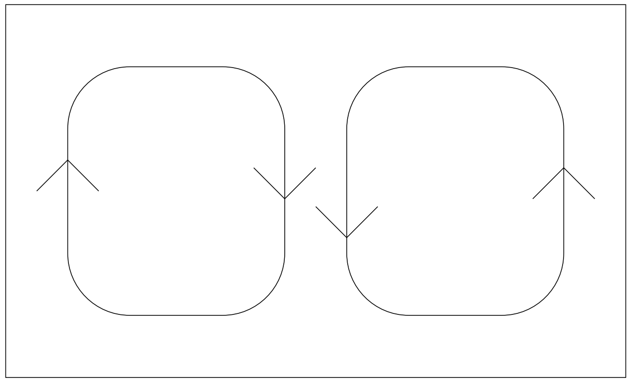
$$E_z = A\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

(leaving out the factor $\exp\left(j\omega t - j\beta z\right)$) which satisfies the $E_{\parallel} = 0$ boundary condition. The transverse magnetic field is then: H_y is the x-derivative of E_z, whereas H_x is the y-derivative of E_z, so we get

$$H_x = A'\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$$

$$H_y = A''\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

We can see from this that the fiels are parallel to the wall of the waveguide, which satisfies the $B_{\perp} = 0$ boundary condition. And there are m cells in the x-direction, and n cells in the y-direction, so it looks like this:



For the TE mode the axial magnetic field looks like

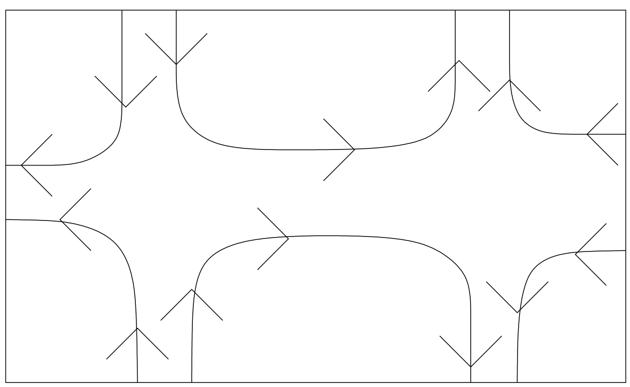
$$H_z = A\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$$

Now E_x is the y-derivative of H_z, and E_y is the x-derivative of H_z, so we get

$$E_x = A'\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

$$E_y = A''\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$$

From this we can see that the electric field is perpendicular to the waveguide wall, which satisfied $E_{\|} = 0$, and that there are two cells in the x-direction and one cell in the y-direction. However they do not look like the cells in the TM mode. Rather, they are split in the middle, looking like this:



(6) What is the cutoff frequency for the TM$_{21}$ mode in a waveguide with dimension $a = 5\,\mathrm{cm}$ and $b = 3\,\mathrm{cm}$.

The critical frequency for a mode mn in a waveguide of dimension a by b is

$$\omega_{c,nm} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

Using given quantities we get

$$\begin{aligned}\omega_{c,21} =& \frac{1}{\sqrt{4\times\pi\times10^{-7}\times8.854\times10^{12}}}\sqrt{\left(\frac{2\pi}{0.05}\right)^2 + \left(\frac{1\pi}{0.03}\right)^2}\\ =& 4.90\times10^{10}\,\mathrm{rad/s}\end{aligned}$$

or in Hz,

$$f_{c,21} = \frac{\omega_{c,nm}}{2\pi} = 7.80\,\mathrm{GHz}$$

(7) For the same waveguide, what is the wavelength of a wave at the cutoff frequency? What is the wavelength of a wave at twice the cutoff frequency?
The expression for wavelength is

$$\lambda_{mn} = \frac{\lambda}{\sqrt{1 - \left(\frac{\omega_{c,mn}}{f}\right)^2}}$$

where λ is the corresponding free-space plane wave wavelength,

$$\lambda = \frac{v}{f}$$

which can also be written

$$\lambda = \frac{2\pi}{\sqrt{\mu\epsilon}\omega}$$

After writing down all these equations we realize that the denominator is zero, such that the wavelength is infinte. This actually makes sense, because at the critical frequency the phase velocity is also infinite, which results in a infinite wavelength for a finite frequency. Recall

$$v = \lambda f$$

At twice the critical frequency we have the free-space wavelength

$$\lambda = \frac{2\pi}{\sqrt{\mu\epsilon}2\omega_c} = \frac{2\pi}{\sqrt{4 \times \pi \times 10^{-7} \times 8.854 \times 10^{-12}} \times 2 \times 4.90 \times 10^{10}} = 1.92\,\mathrm{cm}$$

(8) Still for the same waveguide and at twice the critical frequency, if the waveguide is empty, and the amplitude of $\boldsymbol{E_x}$ is 1 V/m, what is the amplitude of $\boldsymbol{H_y}$?
We need the wave impedance. It turns out that the problem forgot (perhaps intentionally??) to mention whether we are dealing with a TE or TM wave. In that case we better do both. For TM waves the wave impedance is

$$\eta_{TM} = \sqrt{\frac{\mu}{\epsilon}}\sqrt{1 - \left(\frac{\omega_c}{\omega}^2\right)}$$

whereas for TE mode waves it is

$$\eta_{TE} = \frac{\sqrt{\frac{\mu}{\epsilon}}}{\sqrt{1 - \left(\frac{\omega_c}{\omega}^2\right)}}$$

we are in a vacuum, so $\sqrt{\mu/\epsilon} = 377\,\Omega$. At the twice the critical frequency, the other square root evaluates to 0.866. So we get

$$\eta_{TM} = 377 \times 0.866 = 326 \qquad \eta_{TE} = 377/0.866 = 435$$

Now finally, for a TM wave

$$H_y = \frac{E_x}{\eta_{TM}} = \frac{1}{326} = 0.0307\,\mathrm{A/m}$$

whereas for a TE wave

$$H_y = \frac{E_x}{\eta_{TE}} = \frac{1}{435} = 0.00230\,\mathrm{A/m}$$

(9) Consider two dielectric regions with $\boldsymbol{\epsilon_{1r} = 1}$, $\boldsymbol{\epsilon_{2r} = 2}$, and $\boldsymbol{\mu_1 = \mu_2 = \mu_0}$ and an incident plane wave with electric field amplitude $\boldsymbol{E_1 = 1\,\mathrm{V/m}}$. What is the amplitude of the reflected electric field? What is the amplitude of the reflected magnetic field?

The reflection coefficient for normal incidence of a wave from region 1 to the interface with region 2 is

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_1 + \eta_2}$$

so we need to find the wave impedances in the two media.

$$\eta_1 = \sqrt{\mu\epsilon} = \sqrt{\frac{\mu_0\epsilon_0}{=}}377\,\Omega$$

$$\eta_2 = \sqrt{\frac{\mu}{\epsilon}} = \frac{377}{\sqrt{\epsilon_{2r}}} = \frac{377}{\sqrt{2}} = 267\,\Omega$$

and the reflection coefficient becomes

$$\Gamma = \frac{267 - 377}{377 + 267} = -0.171$$

So the amplitude of the reflected electric field is

$$E_1^- = 0.171\,\mathrm{V/m}$$

and the amplitude of the reflected magnetic field is

$$H_1^- = \frac{E_1^-}{\eta_1} = \frac{0.171}{377} = 0.000454\,\mathrm{A/m}$$

(10) What is the incident power? What is the reflected power?

The incident power is

$$P_1^+ = \frac{1}{2}E_1^+ H_1^+ = \frac{1}{2}\frac{(E_1^+)^2}{\eta_1} = \frac{1}{377} = 2.65\,\mathrm{mW/m^2}$$

The reflected power is

$$P_1^- = \frac{1}{2}\frac{(E_1^-)^2}{\eta_1} = \frac{0.171^2}{377} = 0.0776\,\mathrm{mW/m}$$

(11) What is Snell's law? What is the brewster angle? What does total internal reflection mean? Derive an expression for the angle of total internal reflection.
Snell's law related the incidence angle (angle from vertical) in two different media. It has the form

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

where $n = c/v = \sqrt{\mu_r \epsilon_r}$
The Brewster angle is the angle at which the reflection coefficient is zero for a parallel wave. A parallel wave is a wave whose electric field is parallel to the incidence plane. And the incidence plane is perpendicular to the interface.
Total internal reflection occurs when a plane wave is incident on an interface from a higher index medium and at a large angle. In that case all of the energy is reflected back into the higher index medium, none of the energy is transmitted. Total internal reflection occurs when the incidence angle is larger than some critical value.
The critical angle of total internal reflection corresponds to an exit angle of 90°. In that case Snell's law becomes

$$n_1 \sin\theta_{1c} = n_2 \sin\,90^\circ = n_2$$

$$\theta_{1c} = \sin^{-1}\frac{n_2}{n_1}$$