EE 434 Electricity and Magnetism, Spring 2009
Homework #1 Solution

4.1 For this problem I will set zero potential to be at infinity. Also, note that the electric field pattern, and thus the potential pattern, outside the shell is the same as if all the charge were concentrated in a point. In that case we can use the formula

$$\Phi(r) = \frac{Q}{4\pi\epsilon_0 r}$$

(a) In this case, $Q = Q_a$, and we get

$$\Phi(b) = \frac{Q_a}{4\pi\epsilon_0 b}$$

(b) In this case, $Q = Q_a + Q_b$, and we get

$$\Phi(b) = \frac{Q_a + Q_b}{4\pi\epsilon_0 b}$$

4.2 The potential from a point charge dq at location $\vec{r}$ is

$$d\Phi(\vec{p}) = \frac{dq}{4\pi\epsilon_0 \left|\vec{p} - \vec{r}\right|}$$

To get the potential from a distribution of charges we need to integrate over that distribution. In this case it is a line charge, λ, such that $dq = \lambda dl$. Since the total charge on the circular loop is Q, we get

$$\lambda = \frac{Q}{2\pi a}$$

(a) To get the potential at a point along the axis, z, we find that

$$\left|\vec{p} - \vec{r}\right| = \sqrt{z^2 + a^2}$$

Writing the integral we now have

$$\begin{aligned}\Phi(z) &= \int_0^{2\pi a} \frac{\lambda dl}{4\pi\epsilon_0\sqrt{z^2 + a^2}} dl \\ &= \frac{\lambda}{4\pi\epsilon_0\sqrt{z^2 + a^2}} \int_0^{2\pi a} dl \\ &= \frac{Q}{4\pi\epsilon_0\sqrt{z^2 + a^2}}\end{aligned}$$

(b) The electric field is the negative of the gradient of the potential, in cylindrical coordinates,

$$\vec{E}(\rho, \phi, z) = -\nabla\Phi = -\frac{\partial\Phi}{\partial\rho}\hat{\rho} - \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\phi} - \frac{\partial\Phi}{\partial z}\hat{z}$$

In order to compute the gradient we need the potential as a function of the three spatial coordinates (in this case cylindrical ρ, ϕ, and z). But we only computed it as a function of z for $\phi = 0$, and $\rho = 0$. However, because of symmetry the electric field should be along the z-axis such that even if we did have it as a function of the other coordinates, the derivatives evaluated on the z-axis should be zero. Thus we simplify to

$$\begin{aligned}\vec{E}\left(\rho = 0, \phi = 0, z\right) &= -\frac{d\Phi}{dz}\hat{z} \\ &= \frac{Q}{4\pi\epsilon_0}\frac{1}{2}\left(z^2 + a^2\right)^{-\frac{3}{2}} 2z\,\hat{z} \\ &= \frac{Q}{4\pi\epsilon_0}\frac{z}{\left(z^2 + a^2\right)^{\frac{3}{2}}}\,\hat{z}\end{aligned}$$

(c) Compute the electric field directly using Coulomb's law. Notice that due to symmetry only the z-axis electric field will contribute. Other components will cancel because of the symmetry of the charge distribution.

$$\vec{E} = \int \frac{dq}{4\pi\epsilon_0 r^2}\hat{r}$$

Next, note that $r = \sqrt{a^2 + z^2}$, and that the axial component is obtained by multiplying the magntiude by

$$\cos\theta = \frac{z}{\sqrt{z^2 + a^2}}$$

Inserting we get

$$\begin{aligned}\vec{E} &= \hat{z}\int \frac{z\,dq}{4\pi\epsilon_0\left(z^2 + a^2\right)^{\frac{3}{2}}} \\ &= \hat{z}\frac{zQ}{4\pi\epsilon_0\left(z^2 + a^2\right)^{\frac{3}{2}}}\end{aligned}$$