EE 434 Electricity and Magnetism, Spring 2009
Homework #4 Solution

4.25

(a) If we have that

$$\vec{B} = \nabla \times \vec{A}$$

everywhere, then it is also true that

$$\int_S \vec{B} \cdot d\vec{s} = \int_S \left(\nabla \times \vec{A}\right) \cdot d\vec{s}$$

According to Stoke's theorem the surface integral of the curl of a vector field is equal to the line integral of that vector field around the edge of that surface,

$$\int_S \left(\nabla \times \vec{A}\right) \cdot d\vec{s} = \int_L \vec{A} \cdot d\vec{l}$$

and thus

$$\int_S \vec{B} \cdot d\vec{s} = \int_L \vec{A} \cdot d\vec{l}$$

(b) The magnetic potential is computed as

$$\vec{A}\left(\vec{r}\right) = \int_V \frac{\vec{J}\left(\vec{r}'\right)}{\left|\vec{r} - \vec{r}'\right|}\, dv'$$

This is complicated, so we will instead use what we learned in part (a) of this problem. We know that the magnetic field inside an infinite solenoid is $\vec{B} = \hat{z}\mu_0 NI$, where N is the number of windings per unit length, and I is the current in the windings. If we do a line integral of $\vec{A}$ at a fixed radius ρ, we get

$$\int_L \vec{A} \cdot d\vec{l} = \int_S \vec{B} \cdot d\vec{s}$$

$$2\pi\rho A_\phi = \int_S \vec{B} \cdot d\vec{s}$$

When $\rho < a$, then the right-hand side of the equation becomes

$$\int_S \vec{B} \cdot d\vec{s} = \pi\rho^2 B_z = \mu_0 NI\pi\rho^2$$

and thus

$$A_\phi = \frac{\mu_0 NI\pi\rho^2}{2\pi\rho} = \frac{\mu_0 NI\rho}{2}$$

When $\rho > a$, then the right-hand side of the equation becomes

$$\int_S \vec{B}\cdot d\vec{s} = \pi a^2 B_z = \mu_0 N I \pi a^2$$

and thus

$$A_\phi = \frac{\mu_0 N I \pi a^2}{2\pi\rho} = \frac{\mu_0 N I a^2}{2\rho}$$

4.30 The mutual inductance is the amount of magnetic flux going through a loop in one of the conductors, caused by the current in the other conductor, divided by the current in the other conductor. In equation form,

$$M = \frac{\psi}{I}$$

Where ψ is the magnetic flux through the rectangular loop, and I is the current in the infinitely long conductor. The magnetic field around the infinitely long conductor is in the azimuthal direction (because of symmetry), and can be found from Ampere's law at distance ρ

$$\int_L \vec{B}\cdot d\vec{l} = \mu_0 \int_S \vec{J}\cdot d\vec{s}$$

$$2\pi\rho B_\phi = \mu_0 I$$

$$B_\phi = \frac{\mu_0 I}{2\pi\rho}$$

Next we compute the flux through the rectangle,

$$\psi = \int_S \vec{B}\cdot d\vec{s}$$

Since the magnetic field is perpendicular to the rectangular loop and is uniform along the z-direction, we can simplify this to

$$\begin{aligned}
\psi =& b\int_d^{d+a} B_\phi \, d\rho \\
=& b\frac{\mu_0 I}{2\pi}\int_d^{d+a} \frac{1}{\rho}\, d\rho \\
=& b\frac{\mu_0 I}{2\pi}\left[\ln\rho\right]_d^{d+a} \\
=& b\frac{\mu_0 I}{2\pi}\ln\frac{d+a}{d} \\
=& b\frac{\mu_0 I}{2\pi}\ln\left(1+\frac{a}{d}\right)
\end{aligned}$$

The mutual inductance is then

$$M = \frac{\psi}{I} = \frac{\mu_0}{2\pi} b \ln\left(1 + \frac{a}{d}\right)$$

3rd problem Compute the amount of energy per unit length, to build up a current, I, in a infinitely long wire with radius a, assuming that the current is uniformly distributed across the cross-section of the conductor.

The amount of energy required to build up this current is equal to the amount of energy stored in the magnetic field. Thus, we need to compute the integral

$$U = \frac{1}{2} \int_V \vec{B} \cdot \vec{H} \, dv$$

For a unit length of the wire. In cylindrical coordinates we write

$$W = \frac{1}{2} \int_0^\infty \int_0^{2\pi} \frac{B^2}{\mu_0} \rho \, d\phi \, d\rho$$

Because of cylindrical symmetry we can simplify to

$$W = \frac{2\pi}{2\mu_0} \int_0^\infty B^2 \rho \, d\rho$$

The current density inside the conductor is

$$\vec{J} = \hat{z} J_z = \hat{z} \frac{I}{\pi a^2}$$

Inside the conductor the magnetic field is thus

$$2\pi\rho H_\phi = J_z \pi \rho^2 = I \frac{\pi \rho^2}{\pi a^2}$$

$$H_\phi = \frac{I}{2\pi} \frac{\rho}{a^2}$$

Outside the conductor the magnetid field is

$$2\pi\rho H_\phi = I$$

$$H_\phi = \frac{I}{2\pi\rho}$$

We now do the energy integral in two pieces,

$$
\begin{aligned}
W =& \frac{\pi}{\mu_0}\left[\int_0^a B^2\rho\,d\rho + \int_a^\infty B^2\rho\,d\rho\right] \\
=& \frac{\pi}{\mu_0}\left[\int_0^a \mu_0\left(\frac{I}{2\pi}\frac{\rho}{a^2}\right)^2\rho\,d\rho + \int_a^\infty \mu_0\left(\frac{I}{2\pi\rho}\right)^2\rho\,d\rho\right] \\
=& \frac{I^2}{4\pi}\left[\int_0^a \frac{\rho^3}{a^4}\,d\rho + \int_a^\infty \frac{1}{\rho}\,d\rho\right] \\
=& \frac{I^2}{4\pi}\left(\left[\frac{\rho^4}{4a^4}\right]_0^a + [\ln\rho]_a^\infty\right)
\end{aligned}
$$

The second term in the last equation is infinite. Thus, it takes infinite amount of energy, even per unit length, to build current I in a infinitely long wire (Note, the energy per unit length is not infinite if the wire has finite length).