

# EE 434 Electricity and Magnetism, Spring 2009

## Homework #5 Solution

### 5.3

(a) The total electric field will be zero every half wavelength beginning at the surface of the perfect conductor. So we need to first find the wavelength in medium 1. It is

$$\lambda = \frac{v}{\nu} = \frac{1}{\nu\sqrt{\epsilon\mu}} = \frac{\eta}{\nu\mu} = \frac{80}{150 \times 10^6 \times 4\pi \times 10^{-7}} = 0.4244 \text{ m}$$

The first node is  $\lambda/2 = 0.21 \text{ m}$  from the node.

(b) The magnetic field first reaches zero  $\lambda/4 = 0.11 \text{ m}$ .

(c) The expression for the magnetic field in region 1 is

$$\begin{aligned}\vec{H}^{\text{tot}}(z, t) &= 2 \frac{E_{m1}^+}{\eta_1} \cos \beta_1 z \cos \omega t \hat{y} \\ &= \frac{2E_{m1}^+}{\eta_1} \cos \left( \frac{2\pi z}{\lambda_1} \right) \cos \omega t \hat{y}\end{aligned}$$

Since the electric field amplitude,  $E_{m1}^+ = 100$ , we get

$$\vec{H}^{\text{tot}}(z, t) = 2 \frac{100}{80} \cos \left( \frac{2 \times \pi z}{0.42} \right) \cos(\omega t) \hat{y} = 2.5 \cos(14.804 \times z) \cos(\omega t) \hat{y}$$

At  $z = 0$  we get

$$\vec{H}^{\text{tot}}(0, t) = 2.5 \cos(\omega t) \hat{y}$$

At  $z = -2 \text{ m}$  we get

$$\vec{H}^{\text{tot}}(-2, t) = -0.587 \cos(\omega t)$$

### 5.5

(a) In air the wavelength is found from

$$c = \lambda\nu$$

$$\lambda = \frac{c}{\nu} = \frac{3 \times 10^8}{20 \times 10^3} = 15 \text{ km}$$

and thus the antenna length is

$$l = \frac{\lambda}{2} = 7.5 \text{ km}$$

In a conductive medium the wavelength is found from (Equation 3.76b)

$$\frac{2\pi}{\lambda} = \beta = \frac{\omega\sqrt{\mu\epsilon}}{\sqrt{2}} \sqrt{\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1}$$

Inserting  $\omega = 2 \times \pi \times 20 \times 10^3 \text{ s}^{-1}$ ,  $\epsilon = 81\epsilon_0$ ,  $\mu = \mu_0$ , and  $\sigma = 4 \text{ S/m}$ . We get

$$\beta = 0.56 \text{ m}^{-1}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.56} = 11.2 \text{ m}$$

and thus the antenna length must be

$$l = \frac{\lambda}{2} = 5.6 \text{ m}$$

In the air the antenna should have a length  $\lambda_{\text{air}}/2 = 7.5 \text{ km}$ , whereas in water the antenna should have a length  $\lambda_{\text{water}}/2 = 5.6 \text{ m}$ .

**(b)** First we compute the amplitude of the electric field which penetrates the surface of the water, and next the depth of attenuation which results in the specified electric field amplitude. The transmission coefficient is

$$T = \frac{2\eta_2}{\eta_1 + \eta_2}$$

$\eta_2$  is complex, but we are really only interested in the amplitude and not the phase of the transmission coefficient. The intrinsic impedance in the water is

$$\eta_2 = \sqrt{\frac{\mu}{\epsilon - j\frac{\sigma}{\omega}}}$$

If one of the two terms in the denominator is much smaller than the other we can ignore it. We have  $\epsilon = 81 \times 8.854 \times 10^{-12} = 7.172 \times 10^{-10}$ , and  $\sigma/\omega = 4/2 \times \pi \times 20 \times 10^3 = 3.183 \times 10^{-5}$ , about 5 orders of magnitude larger. Clearly we can ignore the first term, and we can write the intrinsic impedance in region 2 as

$$\eta_2 = \sqrt{\frac{\mu}{-j\frac{\sigma}{\omega}}} = \sqrt{\frac{j\mu\omega}{\sigma}} = \sqrt{j} \sqrt{\frac{\mu\omega}{\sigma}} = \left( \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}} \right) \sqrt{\frac{\mu\omega}{\sigma}}$$

Writing  $\eta_2 = \eta_{2r} + j\eta_{2i}$ , we get

$$\eta_{2r} = \eta_{2i} = \frac{1}{\sqrt{2}} \sqrt{\frac{\mu\omega}{\sigma}} = \frac{1}{\sqrt{2}} \sqrt{\frac{4 \times \pi \times 10^{-7} \times 2 \times \pi \times 20 \times 10^3}{4}} = 0.140 \Omega$$

Since

$$\eta_1 = \eta_{\text{air}} = \sqrt{\frac{\mu}{\epsilon}} = 377 \Omega$$

we can see that  $\eta_2 \ll \eta_1$ , so we can further simplify

$$T = \frac{2\eta_2}{\eta_1}$$

and its magnitude is

$$|T| = \frac{2\sqrt{\eta_{2r}^2 + \eta_{2i}^2}}{\eta_1} = \frac{2\sqrt{2}\eta_{2r}}{\eta_1} = \frac{2^{\frac{3}{2}} \times 0.140}{377} = 0.00105$$

In other words, only about 0.1% of the electric field penetrates the surface. The second part of the problem is to find the depth at which 0.01% of the electric field penetrates. The electric field amplitude decreases as a function of distance into the water,  $z$ , as

$$E_m(z) = E_{m0} \exp(-\alpha z)$$

where according to Equation 3.76a

$$\alpha = \frac{\omega\sqrt{\mu\epsilon}}{\sqrt{2}} \sqrt{\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1}$$

Inserting  $\omega = 2 \times \pi \times 20 \times 10^3 = 126 \times 10^3 \text{ s}^{-1}$ ,  $\mu = 4 \times \pi \times 10^{-7} \frac{\text{H}}{\text{m}}$ ,  $\epsilon = 81 \times 8.854 \times 10^{-12} \frac{\text{F}}{\text{m}}$ , and  $\sigma = 4 \frac{\text{S}}{\text{m}}$ , we get

$$\alpha = 0.562 \text{ m}^{-1}$$

which, not surprisingly, is very similar to  $\beta$ . So we solve for  $z$ , when

$$0.01 \times 10^{-2} = T \exp(-\alpha z)$$

$$z = -\frac{1}{\alpha} \ln\left(\frac{10^{-4}}{T}\right) = \frac{1}{0.56} \ln\left(\frac{0.00105}{10^{-4}}\right) = 4.20 \text{ m}$$

In other words, only  $10^{-4}$  of the electric field emitted above the water makes it to 4 m below the surface. This does not bode well for submarine communications. As the frequency increases, the amount of energy that penetrates the surface increases. However, for larger frequencies the depth of penetration decreases. There is thus probably an optimal frequency at which a certain fraction of the electric field penetrates the deepest.

**5.14** This is a case of designing a quarter wave plate. From section 5.7 we know that

$$\epsilon_2 = \sqrt{\epsilon_1 \epsilon_3} \qquad d_2 = \frac{\lambda_2}{4}$$

For the permittivity we find

$$\epsilon_2 = \sqrt{\epsilon_1 \epsilon_3} = \sqrt{\epsilon_0 \times 2.76 \times \epsilon_0} = 1.66\epsilon_0$$

for the wavelength we know that

$$\frac{1}{\sqrt{\epsilon\mu}} = \lambda\nu$$

and thus

$$d_2 = \frac{\lambda}{4} = \frac{1}{4\nu\sqrt{\epsilon\mu}} = \frac{1}{4 \times 10^6 \times \sqrt{1.66 \times 8.854 \times 10^{-12} \times 4 \times \pi \times 10^{-7}}} = 58.2 \text{ m}$$

That is a very thick transformer.

**5.19** The procedure is as follows. We know the impedance in region 3. It is  $Z_3 = \eta_3$ . That is also equal to the impedance at the 2-3 interface in region 2. From that we obtain the reflection coefficient (the same point on the Smith chart). We then rotate that point by one half wavelength. That gives us the total reflection coefficient at the 1-2 interface. It also gives us the impedance, which is also the total impedance in region 1. We convert that to a reflection coefficient in region 1 and find that it is zero. Voila! Here are the steps written out more carefully, with the Smith chart below it to follow along.

1. We begin by noticing that in region 3,  $Z_3 = \eta_3$  because we only have a forward traveling wave. The normalized impedance in region 3 is  $z_3 = Z_3/\eta_3 = 1$ . Marked as point 1.
2. In region 2 at the 2-3 interface the impedance is the same as in region 3,  $Z_2(0) = Z_3(0)$ . The normalized impedance in region 2 is  $z_2 = Z_2/\eta_2 = Z_3/\eta_2 = z_3\eta_3/\eta_2$ . I will choose  $\eta_3/\eta_2 = 4$ . The transformation from  $z_3$  to  $z_2$  is marked as line 2, and  $z_2$  is marked as point 3.
3. The reflection coefficient appears to be roughly  $\Gamma_2(0) = 0.6$ . We use that to translate to get  $\Gamma_2(-\lambda_2/2)$ . We rotate by half a wavelength, which is a full turn around the chart in the clockwise direction. This is marked by circle 4, and the reflection coefficient/impedance at the 1-2 interface is point 5.
4. Next we note that  $Z_1 = Z_2$ , and  $z_1 = Z_2/\eta_1 = z_2\eta_2/\eta_1$ . Since region 1 and 3 are identical, we note that  $\eta_1 = \eta_3$ , so  $z_1 = z_2\eta_2/\eta_3 = 0.25z_2 = 1$ . This transformation is shown as line 6, and the impedance in region 1 at the 1-2 interface is point 7.
5. Finally we read off the reflection coefficient in region 1 at the 1-2 interface and see that it is zero.

# The Complete Smith Chart

## Black Magic Design

