EE 434 Electricity and Magnetism, Spring 2009
Homework #6 Solution

6.1 The first thing I want to do is to verify that this is really a transverse wave. It must tbe true that $\vec{H}$ is perpendicular, $\vec{\beta}$,

$$(4, 3.1, 5 + j) \cdot (-2.28, 3.04) = -9.12 + 9.424 = 0.304$$

This is close enough.
(a) The wavelength is

$$\lambda = \frac{2\pi}{\beta}$$

and $\beta = \sqrt{\beta_x^2 + \beta_y^2 + \beta_z^2} = \sqrt{2.28^2 + 3.04^2} = 3.80$, so

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{3.80} = 1.65$$

The unit vector in the direction of propagation is simply

$$\hat{\beta} = \frac{\beta}{\beta} = \frac{(-2.28, 3.04, 0)}{3.8} = (-0.60, 0.8, 0)$$

(b) The relationship between the electric and magnetic field is

$$\vec{H} = \frac{\hat{\beta} \times \vec{E}}{\eta}$$

$$\hat{\beta} \times \vec{H} = \frac{\hat{\beta} \times \left(\hat{\beta} \times \vec{E}\right)}{\eta}$$

$$\eta\, \hat{\beta} \times \vec{H} = \hat{\beta} \times \left(\hat{\beta} \times \vec{E}\right) = -\vec{E}$$

$$\begin{aligned}
\vec{E} =& \eta\, \vec{H} \times \hat{\beta} \\
=& \eta \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ H_x & H_y & H_z \\ \beta_x & \beta_y & \beta_z \end{vmatrix} \\
=& \hat{x}\eta\left(H_y\beta_z - H_z\beta_y\right) + \hat{y}\eta\left(H_z\beta_x - H_x\beta_z\right) + \hat{z}\eta\left(H_x\beta_y - H_y\beta_x\right)
\end{aligned}$$

Now inserting numbers we get (with $\eta = 377\,\Omega$,

$$\begin{aligned}
\vec{E} =& 377\left(-(5 + j) \cdot 0.8, (5 + j) \cdot (-0.6), 4 \cdot 0.8 - 3.1 \cdot (-0.6)\right) \\
=& 377\left(-4 - 0.8j, -3 - 0.6j, 5.06\right)
\end{aligned}$$

6.2

(a) The electric field vector must be perpendicular to the direction of propagation. Thus,

$$(E_x, E_y, E_z) \cdot (\beta_x, \beta_y, \beta_z) = 0$$

so we need to determine β_x to satisfy that

$$E_x\beta_x + E_y\beta_y + E_z\beta_z = 0$$

$$\begin{aligned}\beta_x &= -\frac{E_y\beta_y + E_z\beta_z}{E_x} \\ &= -\frac{0.5 \cdot 1.8 + 5 \cdot (-2.1)}{2} \\ &= 4.8\end{aligned}$$

The direction of propagation is thus

$$\vec{\beta} = (4.8, 1.8, -2.1)$$

or as a unit vector

$$\hat{\beta} = \frac{(4.8, 1.8, -2.1)}{\sqrt{4.8^2 + 1.8^2 + 2.1^2}} = (0.866, 0.325, -0.379)$$

(b) The wavelength is related to β by

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{4.8^2 + 1.8^2 + 2.1^2}} = 1.134$$

(c) The vector magnetic field intensity, $\vec{H}$, is related to the electric field by

$$\begin{aligned}\vec{H} &= \frac{\hat{\beta} \times \vec{E}}{\eta} \\ &= \frac{1}{\eta}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \beta_x & \beta_y & \beta_z \\ E_x & E_y & E_z \end{vmatrix} \\ &= \frac{1}{\eta}\hat{x}\left(\beta_y E_z - \beta_z E_y\right) + \frac{1}{\eta}\hat{y}\left(\beta_z E_x - \beta_x E_z\right) + \frac{1}{\eta}\hat{z}\left(\beta_x E_y - \beta_y E_x\right)\end{aligned}$$

Inserting values we get

$$\begin{aligned}\vec{H} &= \frac{1}{377}\left(0.325 \cdot 5 + 0.379 \cdot 0.5, -0.379 \cdot 2 - 0.866 \cdot 5, 0.866 \cdot 0.5 - 0.325 \cdot 2\right) \\ &= \frac{1}{377}\left(1.815, -5.080, -0.217\right)\end{aligned}$$

6.9 The spatial variation of the incident electric field is

$$\vec{E}^i = 25\,(0.866\,\hat{x} - 0.5\hat{z})\exp(-5j\,x - 8.66j\,z)$$

(a) The frequency is found from

$$v = \lambda\nu = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon}}$$

$$\omega = \frac{\beta}{\sqrt{\mu\epsilon}} = \frac{\sqrt{5^2 + 8.66^2}}{\sqrt{4\times\pi\times 10^{-7}\times 4\times 8.854\times 10^{-12}}} = 1.4990\times 10^9\,\mathrm{s}^{-1}$$

The real-time form of the incident field is

$$\vec{E}^i = \vec{E}^i_o \cos(\omega t - \vec{\beta}\cdot\vec{r})$$

Inserting everything we know we get

$$\vec{E}^i = 25\;(0.866\,\hat{x} - 0.5\,\hat{z})\cos\left(1.4990\times 10^9\times t - 5x - 8.66z\right)$$

(b) The procedure is as follows: (1) determine the incidence angle, transmission angle, and wave impedance in the two media, (2) compute the amplitude of the incident field, the transmission coefficient, and from those the amplitude of the transmitted field, (3) compute the vector electric field from the transmission angle and the ampliutde.
(1) The amplitude of the incident electric field is

$$E^i_o = 25\times(0.866^2 + 0.5^2) = 25$$

(2) The incidence angle is

$$\theta_i = \sin^{-1}\frac{\beta_x}{\beta} = \sin^{-1}\frac{5}{10} = 30°$$

(3) The impedances are

$$\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}} = \sqrt{\frac{4\pi\times 10^{-7}}{4\times 8.854\times 10^{-12}}} = 188.37\,\Omega$$

$$\eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}} = \sqrt{\frac{4\pi\times 10^{-7}}{10\times 8.854\times 10^{-12}}} = 119.13\,\Omega$$

(4) The transmission angle is

$$\theta_t = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_i\right) = \sin^{-1}\left(\sqrt{\frac{4}{10}}\sin\,30°\right) = 18.435°$$

(5) The transmission coefficient is

$$\tau = \frac{2\eta_2\cos\theta_i}{\eta_1\cos\theta_i + \eta_2\cos\theta_t} = \frac{2\cdot 119.13\cdot\cos\,30°}{188.37\cdot\cos\,30° + 119.13\cdot\cos\,18.435°} = 0.74720$$

(6) The amplitude of the transmitted electric field is

$$E_o^t = E_o^i\,\tau = 25 \cdot 0.74720 = 18.680$$

(7) The components of the transmitted electric field are

$$\begin{aligned} E_{ox}^t &= E_o^t \sin\theta_t = 18.68 \,\sin\, 18.43^\circ = 5.905\,\hat{x} \\ E_{oz}^t &= -E_o^t \cos\theta_t = -18.68 \,\cos\, 18.43^\circ = -17.72\,\hat{x} \end{aligned}$$

(8) The incident wave number is

$$\beta^i = \sqrt{5^2 + 8.66^2} = 10$$

(9) The transmitted wave number is found from

$$v^t = \frac{\omega}{\beta^t}$$

$$\frac{v^t}{v^i} = \frac{\beta^i}{\beta^t} = \frac{c/n_2}{c/n_1}$$

$$\frac{n_1}{n_2} = \frac{\beta^i}{\beta^t}$$

$$\beta^t = \frac{n_2}{n_1}\beta^i = \sqrt{\frac{\epsilon_2}{\epsilon_1}}\beta^i = \sqrt{\frac{10}{4}} \cdot 10 = 15.81\,\mathrm{m}^{-1}$$

(10) The components of the wave number is

$$\begin{aligned} \beta_x^t =& \beta^t \sin\theta_t = 15.81 \times \sin\, 18.435^\circ = 5.00 \\ \beta_z^t =& \beta^t \cos\theta_t = 15.81 \times \cos\, 18.435^\circ = 15.00 \end{aligned}$$

(11) The frequency of the wave is

$$\omega = v\,\beta = \frac{c}{n}\beta = \frac{c}{\sqrt{\epsilon}}\beta = \frac{c}{\sqrt{\epsilon_i}}\beta_i = \frac{3 \times 10^8}{\sqrt{4}} 10 = 1.50 \times 10^9\,\mathrm{s}^{-1}$$

(12) Finally we can write the real-time form of the transmitted electric field,

$$\begin{aligned} E^t(x,y,z,t) &= \left(E_{ox}^t \hat{x} + E_{oz}^t \hat{z}\right) \cos(\omega t - \beta_x^t x - \beta_z^t z) \\ &= (5.905\hat{x} - 17.72\hat{z}) \cos(1.5 \times 10^9\, t - 5\, x - 15\, z) \end{aligned}$$

6.11
(a)

$$\theta_B = \tan^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}} = \tan^{-1}\sqrt{6} = 67.79^\circ$$

(b) I am assuming that the positive direction for the electic field is upward, which is in the negative z-direction. In that case,

$$E(x, y, z, t) = \left(E^i_{ox}\hat{x} + E^i_{oz}\hat{z}\right)\cos(\omega t - \beta^i_x\, x - \beta^i_z\, z)$$

we have

$$\beta^i = \frac{2\pi}{\lambda_i} = \frac{2\pi}{0.6283 \times 10^{-6}} = 1.000 \times 10^7\,\mathrm{s}^{-1}$$

And

$$\beta^i_x = \beta^i \sin\theta_i = 1.000 \times 10^7 \times \sin\, 67.79° = 9.258 \times 10^6\,\mathrm{s}^{-1}$$

$$\beta^i_z = \beta^i \cos\theta_i = 1.000 \times 10^{-7} \times \cos\, 67.79° = 3.780 \times 10^6\,\mathrm{s}^{-1}$$

$$E^i_{oz} = -E^i_o \sin\theta_i = -5 \times \sin\, 67.79° = -4.629\,\mathrm{V/m}$$

$$E^i_{ox} = E^i_o \cos\theta_i = 5 \times \cos\, 67.79° = 1.890\,\mathrm{V/m}$$

$$\omega = c\beta_i = 3.000 \times 10^8 \times 1.000 \times 10^7 = 3.000 \times 10^{15}\,\mathrm{s}^{-1}$$

Inserting those values we get

$$E(x, y, z, t) = (1.890\,\hat{z} - 4.629\,\hat{x})\cos(3 \times 10^{15}\,t - 9.258 \times 10^6\,x - 3.780 \times 10^6\,z)$$

(c) The transmitted field has the same amplitude as the incident field, $E^t_o = E^i_o$. The transmission angle is found from Snell's law

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

Because $\mu = \mu_0$, $n = \sqrt{\epsilon}$, so we get

$$\theta_t = \sin^{-1}\left(\sqrt{\frac{\epsilon_1}{\epsilon_2}}\sin\theta_i\right) = \sin^{-1}\left(\frac{1}{\sqrt{6}}\sin 67.79°\right) = 22.207°$$

The transmitted electric field is

$$\begin{aligned}\vec{E}^t_o =& E^t_{ox}\,\hat{x} + E^t_{oz}\,\hat{z} = E^i_o \sin\,\theta_t\,\hat{x} - E^i_o \cos\,\theta_t\,\hat{z} = 5 \times \sin\, 22.207°\,\hat{x} - 5 \times \cos\, 22.207°\,\hat{z} \\ =& 1.890\,\hat{x} - 4.629\,\hat{z}\end{aligned}$$

(d) The critical angle is the incidence angle for which the exit angle is 90°. Thus, from Snell's law,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

gives

$$\begin{aligned}\theta_2 &= \sin^{-1}\left(\frac{n_1}{n_2}\sin\ 90^\circ\right)\\ &= \sin^{-1}\left(\frac{1}{\sqrt{\epsilon_2}}\right) = \sin^{-1}\left(\frac{1}{\sqrt{6}}\right) = 24.095^\circ\end{aligned}$$