EE 434 Electricity and Magnetism, Spring 2009
Homework #7 Solution

(a) The reflection coefficient for the fields for normal incidence is

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_1 + \eta_2}$$

The wave impedances are

$$\eta = \sqrt{\frac{\mu}{\epsilon}}$$

Since $\mu = \mu_0$ everywhere, they cancel out. Also, ϵ_0 factors cancel out, and we get

$$\Gamma = \frac{\frac{1}{\sqrt{\epsilon_{2r}}} - \frac{1}{\sqrt{\epsilon_{1r}}}}{\frac{1}{\sqrt{\epsilon_{1r}}} + \frac{1}{\sqrt{\epsilon_{2r}}}} = \frac{\sqrt{\epsilon_{1r}} - \sqrt{\epsilon_{2r}}}{\sqrt{\epsilon_{1r}} + \sqrt{\epsilon_{2r}}} = \frac{1-\sqrt{5}}{1+\sqrt{5}} = -0.3820$$

(b) Because the poynting vector represents energy flow, and the reflection coefficient is (of course) the same for electric and magnetic fields, the power reflection coefficient is the square of the field reflection coefficient, so

$$\Gamma_{\text{power}} = \Gamma^2 = 0.3820^2 = 0.1459$$

(c) We have a forward traveling wave and a reverse traveling wave of the same amplitude. At the conductor the electric fields exactly cancel, whereas the magnetic fields add. Moving away from the conductor the total field repeats with a periodicity of one half wavelength. So one quarter wavelength from the conductor the electric field amplitude is twice the incoming field amplitude. One half wavelength from the conductor the electric field amplitude is always zero. For the magnetic field it is the opposite: one half wavelength from the conductor the amplitude is twice the amplitude of the incoming wave, whereas one quarter wavelength from the conductor the magnetic field is zero. If the incoming electric and magnetic field amplitudes are E_o^+ and H_o^+ respectively, we can write the total field amplitudes as

$$E_o = \left|2E_o^+ \sin\left(\frac{2\pi\, z}{\lambda}\right)\right| \qquad H_o = \left|2H_o^+ \cos\left(\frac{2\pi\, z}{\lambda}\right)\right|$$

(The absolute values because we are really just being asked for the amplitude). Next we need the wavelength and the magnetic field amplitude. Since we are in air, the speed of the wave is the speed of light, so

$$\lambda = \frac{c}{\nu} = \frac{3 \times 10^8}{10^8} = 3\,\text{m}$$

Now we are ready to insert values

$$E_o(1\,\text{m}) = \left|2 \times 5 \times \sin\left(\frac{2\pi}{3}\right)\right| = 8.66\,\text{V/m}$$

$$E_o(2\,\text{m}) = \left|2 \times 5 \times \sin\left(\frac{4\pi}{3}\right)\right| = 8.66\,\text{V/m}$$

$$E_o(3\,\mathrm{m}) = \left|2 \times 5 \times \sin\left(\frac{6\pi}{3}\right)\right| = 0$$

(d) The magnetic fiels amplitude is

$$H_o = \frac{E_o}{\eta} = E_o\sqrt{\frac{\epsilon_o}{\mu_o}} = 5\sqrt{\frac{8.854 \times 10^{-12}}{4\pi \times 10^{-7}}} = 0.01326\,\frac{\mathrm{A}}{\mathrm{m}}$$

$$H_o(1\,\mathrm{m}) = \left|2 \times 0.01326 \cos\left(\frac{4\pi}{3}\right)\right| = 0.01326\,\mathrm{A/m}$$

$$H_o(2\,\mathrm{m}) = \left|2 \times 0.01326 \cos\left(\frac{4\pi}{3}\right)\right| = 0.01326\,\mathrm{A/m}$$

$$H_o(3\,\mathrm{m}) = \left|2 \times 0.01326 \cos\left(\frac{6\pi}{3}\right)\right| = 0.02652\,\mathrm{A/m}$$

(e) The quarter-wave plate is a layer which is inserted between two media of different properties. It's purpose is to act as a anti-reflective layer which eliminates a reflected wave in medium 1, which would otherwise appear because of the impedance mismatch between the two media. It turns out that a layer of a quarter wavelength which has wave impedance $\eta = \sqrt{\eta_1 \eta_2}$ cause the amplitude of the reflected wave in medium 1 to be zero.
(f) The quarter wave plate must have

$$\epsilon_r = \sqrt{\epsilon_{1r}\epsilon_{2r}} = \sqrt{1.5 \times 5} = 2.739$$

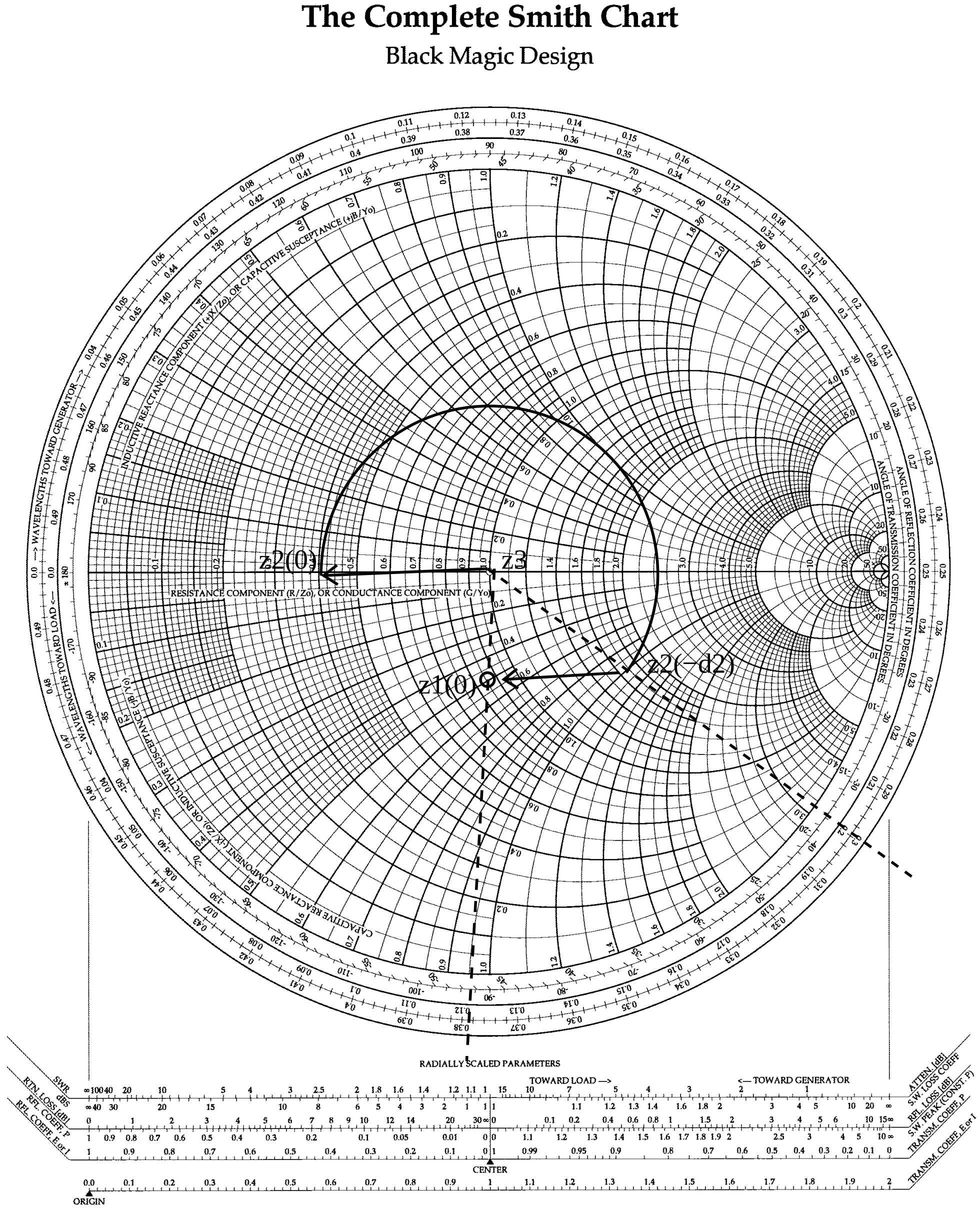
Its thickness must be one quarter wavelength,

$$\begin{aligned} d =& \frac{\lambda}{4} = \frac{v}{4\nu} = \frac{1}{\sqrt{\epsilon\mu} \times 4\nu} = \frac{1}{\sqrt{\epsilon_r\epsilon_0\mu_0} \times 4\nu} \\ =& \frac{1}{\sqrt{2.739 \times 8.854 \times 10^{-12} \times 4 \times \pi \times 10^{-7}} \times 4 \times 100 \times 10^6} = 0.453\,\mathrm{m} \end{aligned}$$

(g) A half-wave plate is a layer which is inserted between two media of the same properties. If the layer is exactly one half wavelength thick it turns out that no wave is reflected off of it, regardless of its dielectric properties. It is often used when a wall is needed between two different equal regions, which is transparent to a particular wavelength. A good example is the dome surrounding a radar.
(h) The nature of the two media is irrelevant as long as they are the same. And we can choose the dielectric properties of the layer arbitrarily, so I will choose $\epsilon_r = 3$. In that case the thickness should be

$$\begin{aligned} d =& \frac{1}{\sqrt{\epsilon_r\epsilon_0\mu_0} \times 2\nu} \\ =& \frac{1}{\sqrt{3 \times 8.854 \times 10^{-12} \times 4 \times \pi \times 10^{-7}} \times 2 \times 10 \times 10^9} \\ =& 8.65\,\mathrm{mm} \end{aligned}$$

(i) We proceed as follows. We know that $Z_3 = \eta_3$, and of course that $z_3 = 1$. That point is the center of the Smith chart as labeled. Next we know that $Z_2 = Z_3$, which means that $\eta_2 z_2 = \eta_3 z_3 = \eta_3$, and thus $z_2 = \frac{\eta_3}{\eta_2} = \frac{20}{50} = 0.4$. That point is labeled on the Smith chart as well. That same point is the reflection coefficient at the 2-3 interface. Next we rotate it 0.8 waves toward the generator. That is one turn and then a partial turn corresponding to 0.3 waves. We show the 0.3 wave turn. The end point is $z_2(-d_2) = (1.7, -1.0)$ at the 1-2 interface. Next we know that $Z_1(0) = Z_2(-d_2)$, such that $z_1(0) = z_2(-d_2)\frac{\eta_2}{\eta_1} = z_2(-d_2) \times 0.5 = (0.85, 0.50)$. That point is labeled on the Smith chart as $z_1(0)$. We can then read off the reflection coefficient. Roughly it appears to be $\Gamma_1 = 0.2e^{-j92^\circ}$



(j) The brewster angle (the angle of zero reflection of waves in which the E-field is parallel to the plane of incidence). It is

$$\tan\theta_B = \sqrt{\frac{\epsilon_2}{\epsilon_1}}$$

Remember that $n = \sqrt{\epsilon_r \mu_r}$, so that

$$\tan\theta_B = n$$

$$\theta_B = \tan^{-1} n = \tan^{-1} 1.6 = 58.0^\circ$$

(k) We use Snell's law,

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

$$\begin{aligned}\theta_t &= \sin^{-1}\left(\frac{1}{n_2}\sin\theta_i\right)\\ &= \sin^{-1}\left(\frac{1}{1.6}\sin\ 45^\circ\right)\\ &= 26.23^\circ\end{aligned}$$

(l) For the parallel case the reflection and transmission coefficients are

$$\Gamma_\parallel = \frac{\cos\theta_t - \sqrt{\frac{\epsilon_2}{\epsilon_1}}\cos\theta_i}{\cos\theta_t + \sqrt{\frac{\epsilon_2}{\epsilon_1}}\cos\theta_i}$$

$$\tau_\parallel = \frac{2\cos\theta_i}{\cos\theta_t + \sqrt{\frac{\epsilon_2}{\epsilon_1}}\cos\theta_i}$$

Inserting values we get for reflection

$$\Gamma_\parallel = \frac{\cos 26.23^\circ - 1.6\cos 45^\circ}{\cos 26.23^\circ + 1.6\cos 45^\circ} = -0.1155$$

and thus a reflected electric field amplitude of

$$E^r = 5 \times 0.1155 = 0.5777\,\mathrm{V/m}$$

And for transmission we get

$$\tau_\parallel = \frac{2\cos 45^\circ}{\cos 26.23^\circ + 1.6\cos 45^\circ} = 0.6972$$

and thus a transmitted electric field amplitude of

$$E^t = 5 \times 0.6972 = 3.486\,\mathrm{V/m}$$

For the perpendicular case the reflection and transmission coefficients are

$$\Gamma_\perp = \frac{\cos\theta_i - \sqrt{\frac{\epsilon_2}{\epsilon_1}}\cos\theta_t}{\cos\theta_i + \sqrt{\frac{\epsilon_2}{\epsilon_1}}\cos\theta_t}$$

$$\tau_\perp = \frac{2\cos\theta_i}{\cos\theta_i + \sqrt{\frac{\epsilon_2}{\epsilon_1}}\cos\theta_t}$$

Inserting values we get

$$\Gamma_\perp = \frac{\cos 45^\circ - 1.6\cos 26.23^\circ}{\cos 45^\circ + 1.6\cos 26.23^\circ} = -0.3399$$

and thus the reflected electric field amplitude is

$$E^r = 5 \times 0.3399 = 1.699\,\mathrm{V/m}$$

And for tranmssion we get

$$\tau_\perp = \frac{2\cos 45^\circ}{\cos 45^\circ + 1.6\cos 26.23^\circ} = 0.6601$$

and thus the transmitted electric field amplitude is

$$E^t = 5 \times 0.6601 = 3.300\,\mathrm{V/m}$$