EE 434 Electricity and Magnetism, Spring 2009
Homework #9 Solution

8.1
The cutoff frequency for both TE and TM modes are

$$\omega_{c,nm} = \frac{1}{\sqrt{\mu\epsilon}} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

The lowest mode is either the TE_{10} or TE_{01} mode. Notice that $a > b$, so it is the $m = 1$, $n = 0$ mode which has the smallest frequency. The factor,

$$\omega_0 = \frac{1}{\sqrt{\mu_0\epsilon_0}}\sqrt{\left(\frac{\pi}{a}\right)} = \frac{1}{\sqrt{4\pi \times 10^{-7} \times 8.854 \times 10^{-12}}}\sqrt{\left(\frac{\pi}{2.3 \times 10^{-2}}\right)^2} = 4.095 \times 10^{10}\,\text{Hz}$$

such that

$$\omega_{c,10} = \frac{\omega_0}{\sqrt{\epsilon_r\mu_r}}$$

(a) For $\epsilon_r = 1$, $\mu_r = 1$, we get

$$\omega_{c,10} = \omega_0 = 40.95\,\text{GHz}$$

(b) For $\epsilon_r = 4$, and $\mu_r = 1$, we get

$$\omega_{c,10} = \frac{\omega_0}{\sqrt{4}} = 10.24\,\text{GHz}$$

(c) For $\epsilon_r = 81$, and $\mu_r = 1$, we get

$$\omega_{c,10} = \frac{\omega_0}{\sqrt{81}} = 4.55\,\text{GHz}$$

8.5 The axial magnetic field is (from equation 8.34)

$$H_z(x, y, z, t) = A\cos\left(\frac{m\pi}{a}\right)\cos\left(\frac{n\pi}{b}\right)e^{j\omega t - j\beta z}$$

whereas the expressions for the transverse electric and magnetic fields are (from equation 8.4)

$$\begin{aligned}
E_x &= -\frac{1}{\gamma^2 + \omega^2\mu\epsilon}\left(\gamma\frac{\partial E_z}{\partial x} + j\omega\mu\frac{\partial H_z}{\partial y}\right) \\
E_y &= \frac{1}{\gamma^2 + \omega^2\mu\epsilon}\left(j\omega\mu\frac{\partial H_z}{\partial x} - \gamma\frac{\partial E_z}{\partial y}\right) \\
H_x &= \frac{1}{\gamma^2 + \omega^2\mu\epsilon}\left(j\omega\epsilon\frac{\partial E_z}{\partial y} - \gamma\frac{\partial H_z}{\partial x}\right) \\
H_y &= -\frac{1}{\gamma^2 + \omega^2\mu\epsilon}\left(j\omega\epsilon\frac{\partial E_z}{\partial x} + \gamma\frac{\partial H_z}{\partial y}\right)
\end{aligned}$$

Now of course, since we are looking at TE modes, $E_z = 0$ which simplifies the calculations, so

$$\begin{aligned} E_x =& -\frac{1}{\gamma^2 + \omega^2\mu\epsilon} j\omega\mu \frac{\partial H_z}{\partial y} \\ =& -\frac{j\omega\mu}{\gamma^2 + \omega^2\mu\epsilon} A\frac{n\pi}{b} \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) e^{j\omega t - j\beta z} \end{aligned}$$

$$\begin{aligned} E_y =& \frac{1}{\gamma^2 + \omega^2\mu\epsilon} j\omega\mu \frac{\partial H_z}{\partial x} \\ =& \frac{j\omega\mu}{\gamma^2 + \omega^2\mu\epsilon} A\frac{m\pi}{a} \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) e^{j\omega t - j\beta z} \end{aligned}$$

$$\begin{aligned} H_x =& -\frac{1}{\gamma^2 + \omega^2\mu\epsilon} \gamma\omega\epsilon \frac{\partial H_z}{\partial x} \\ =& \frac{\gamma}{\gamma^2 + \omega^2\mu\epsilon} A\frac{m\pi}{a} \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) e^{j\omega t - j\beta z} \end{aligned}$$

$$\begin{aligned} H_y =& -\frac{1}{\gamma^2 + \omega^2\mu\epsilon} \gamma \frac{\partial H_z}{\partial y} \\ =& -\frac{\gamma}{\gamma^2 + \omega^2\mu\epsilon} A\frac{n\pi}{b} \sin\left(\frac{m\pi}{a}\right)\cos\left(\frac{n\pi}{b}\right) e^{j\omega t - j\beta z} \end{aligned}$$

The real components are

$$\begin{aligned} E_x =& \frac{\omega\mu}{-\beta^2 + \omega^2\mu\epsilon} A\frac{n\pi}{b} \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\sin(\omega t - \beta z) \\ E_y =& -\frac{\omega\mu}{-\beta^2 + \omega^2\mu\epsilon} A\frac{m\pi}{a} \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\sin(\omega t - \beta z) \\ H_x =& -\frac{\beta}{-\beta^2 + \omega^2\mu\epsilon} A\frac{m\pi}{a} \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\sin(\omega t - \beta z) \\ H_y =& \frac{\beta}{-\beta^2 + \omega^2\mu\epsilon} A\frac{n\pi}{b} \sin\left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi}{b}\right)\sin(\omega t - \beta z) \end{aligned}$$

The real component of the axial magnetic field is

$$H_z = A\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\cos(\omega t - \beta z)$$

8.6 The x-component of the electric field is

$$E_x = A\cos\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi y}{b}\right) sin\left(7\pi \times 10^{10} t - \beta z\right)$$

(a) First, it is clear that $m = 1$, and $n = 2$. The next question is whether this is a TM or TE mode. The expression for E_x is

$$E_x = A'\left(\gamma\frac{\partial E_z}{\partial x} + j\omega\mu\frac{\partial H_z}{\partial y}\right)$$

For a TM mode the axial electric field is a product of two sines. Since E_x comes from a derivative along the the x-coordinate, we would get a cosine variation along the x-direction and a sine variation along the y-direction.

For a TE mode the axial magnetic field is a product of two cosines. Since E_x comes from a derivative along the y-coordinate, we would have a cosine variation along the x-direction and a sine variation along the y-direction.

It could thus be either a TM mode or a TE mode or a combination of the two.

(b) The operating frequency is

$$\omega = 7\pi \times 10^{10}\text{s}^{-1}$$

or

$$f = \frac{\omega}{2\pi} = \frac{7}{2} \times 10^{10}\,\text{Hz} = 35\,\text{GHz}$$

(c) The propagation constant is

$$\begin{aligned}\beta =& \sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \\ =& \sqrt{\left(\frac{7\pi\times 10^10}{3\times 10^8}\right)^2 - \left(\frac{\pi}{2.3\times 10^{-2}}\right)^2 - \left(\frac{2\pi}{1.2\times 10^{-2}}\right)^2} \\ =& 494.5\,\text{m}^{-1}\end{aligned}$$

(d) The cutoff frequency is

$$\begin{aligned}\omega_{c,12} =& \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \\ =& 3\times 10^8\sqrt{\left(\frac{\pi}{2.3\times 10^{-2}}\right)^2 + \left(\frac{2\pi}{1.2\times 10^{-2}}\right)^2} \\ =& 1.623\times 10^{11}\,\text{s}^{-1}\end{aligned}$$

or

$$f_{c,12} = 2.58\times 10^{10}\,\text{Hz} = 25.8\,\text{GHz}$$

The wave impedance depends on whether it is a TM mode or a TE mode. Since we cannot determine that, we report both wave impedances. If it is a TM mode then

$$
\begin{aligned}
\eta_{TM} =& \sqrt{\frac{\mu}{\epsilon}}\sqrt{1-\left(\frac{\omega_{c,nm}}{\omega}\right)^2} \\
=& 377\sqrt{1-\left(\frac{25.8}{35}\right)^2} \\
=& 254.8\,\Omega
\end{aligned}
$$

If it is a TE mode wave then

$$
\begin{aligned}
\eta_{TE} =& \sqrt{\frac{\mu}{\epsilon}}\frac{1}{\sqrt{1-\left(\frac{\omega_{c,nm}}{\omega}\right)}} \\
=& \frac{377}{\sqrt{1-\left(\frac{25.8}{35}\right)^2}} \\
=& 558.0\,\Omega
\end{aligned}
$$

8.10 The expression for the cutoff frequency is

$$
\omega_{c,nm} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}
$$

Since $\epsilon_r = 4$, and $\mu_r = 1$, we have

$$
\frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{\sqrt{4}} = \frac{c}{2}
$$

(a) We are given $\omega_{c,10}$ and $\omega_{c,03}$, so

$$
\omega_{c,10} = \frac{c}{2}\frac{\pi}{a}
$$

so that

$$
\begin{aligned}
a =& \frac{c}{2}\frac{\pi}{\omega_{c,10}} \\
=& \frac{3\times 10^8}{2}\frac{\pi}{2\pi\times 10\times 10^9} \\
=& 0.0075\,\text{m} = 7.5\,\text{mm}
\end{aligned}
$$

and

$$
\omega_{c,03} = \frac{c}{2}\frac{3\pi}{b}
$$

so that

$$
\begin{aligned}
b =& \frac{c}{2}\frac{\pi}{\omega_{c,03}} \\
=& \frac{3\times 10^8}{2}\frac{3\pi}{2\pi \times 60 \times 10^9} \\
=& 0.0038\,\mathrm{m} = 3.8\,\mathrm{mm}
\end{aligned}
$$

(b) From the formula for the cutoff frequency we can see that the cutoff frequency in air must be $\sqrt{\epsilon_r}$ for glass larger than the cutoff frequency for glass, such that

$$f_{c,10,\mathrm{air}} = 2 f_{c,10,\mathrm{glass}} = 2 \times 10\,\mathrm{GHz} = 20\,\mathrm{GHz}$$