EE 434 Electricity and Magnetism, Spring 2009
Homework #10 Solution

8.10c The modes which can be excited are the ones which are supported at a frequency of 65 GHz. In other words, the modes which have a critical frequency less than or equal to 65 GHz. The formula for the critical frequency of both TM and TE modes is

$$f_{c,mn} = \frac{1}{2\pi\sqrt{\mu_0\epsilon_0}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

In homewor set 9 we found that the waveguide dimensions are $a = 0.75$ cm and $b = 0.375$ cm. We can make a table of the critical frequency and then see which frequencies come out less than 65 GHz:

$\downarrow n/m \rightarrow$	0	1	2	3	4
0	0	19.99	39.97	59.96	79.95
1	39.97	44.69	56.53	72.06	89.38
2	79.95	82.41	89.38	99.93	113.06
3	120.92	121.57	126.41	134.07	144.12
4	159.89	161.14	164.81	170.76	178.76

Now remember that TM modes must have both $m > 0$ and $n > 0$, whereas TE modes only need to have one of them non-zero. Valid TE modes are then TE_{01}, TE_{10}, TE_{11}, TE_{20}, TE_{21}, TE_{30}, whereas valid TM modes are TM_{11}, and TM_{21}.

8.12 The lowest TM mode has the axial electric field

$$E_z = A\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{b}y\right)\exp\left(j\omega t - j\beta z\right)$$

The X and Y components of the magnetic field are

$$\begin{aligned} H_x =& \frac{1}{-\beta^2 + \omega^2\mu\epsilon} j\omega\epsilon\frac{\partial E_z}{\partial y} \\ =& \frac{1}{-\beta^2 + \omega^2\mu\epsilon} j\omega\epsilon A\frac{\pi}{b}\sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{b}y\right)\exp\left(j\omega t - j\beta z\right) \end{aligned}$$

$$\begin{aligned} H_y =& -\frac{1}{-\beta^2 + \omega^2\mu\epsilon} j\omega\epsilon\frac{\partial E_z}{\partial x} \\ =& -\frac{1}{-\beta^2 + \omega^2\mu\epsilon} j\omega\epsilon A\frac{\pi}{a}\cos\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{b}y\right)\exp\left(j\omega t - j\beta z\right) \end{aligned}$$

Next we need the real components of the magnetic field,

$$H_x = -\frac{\omega\epsilon A\frac{\pi}{b}}{-\beta^2 + \omega^2\mu\epsilon}\sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{b}y\right)\sin\left(\omega t - \beta z\right)$$

$$H_y = \frac{\omega\epsilon A\frac{\pi}{a}}{-\beta^2 + \omega^2\mu\epsilon}\cos\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{b}y\right)\sin\left(\omega t - \beta z\right)$$

On the $x = 0$ wall,

$$H_x = 0 \qquad H_y = \frac{C}{b} \sin\left(\frac{\pi}{b} y\right) \sin(\omega t - \beta z)$$

On the $x = a$ wall

$$H_x = 0 \qquad H_y = -\frac{C}{b} \sin\left(\frac{\pi}{b} y\right) \sin(\omega t - \beta z)$$

On the $y = 0$ wall

$$H_x = -\frac{C}{a} \sin\left(\frac{\pi}{a} x\right) \sin(\omega t - \beta z) \qquad H_y = 0$$

on the $y = b$ wall

$$H_x = \frac{C}{a} \sin\left(\frac{\pi}{a} x\right) \sin(\omega t - \beta z) \qquad H_y = 0$$

Next, note that since the field inside the waveguide wall is zero, $\Delta \vec{H} = \vec{H}$. The current at the boundary is derived from

$$\vec{J} = \vec{n} \times \vec{H}$$

we can also use the right-hand rule to figure the direction of the current since we know its magnitude. When the fingers are curled around the direction of the current they must, inside the waveguide, point in the direction of the magnetic field.
On the $x = 0$ wall the magntic field points in the positive y-direction, and thus the current points in the positive z-direction, thus

$$J_x = 0 \qquad J_y = 0 \qquad J_z = \frac{C}{b} \sin\left(\frac{\pi}{b} y\right) \sin(\omega t - \beta z)$$

On the $x = a$ wall, the magnetic field points in the negative y-direction, so the current also points in the positive z-direction (because we are now looking at it from the other side)

$$J_x = 0 \qquad J_y = 0 \qquad J_z = \frac{C}{b} \sin\left(\frac{\pi}{b} y\right) \sin(\omega t - \beta z)$$

On the $y = 0$ wall the magnetic field points in the negative x-direction, so the current must point in the positive z-direction,

$$J_x = 0 \qquad J_y = 0 \qquad J_z = \frac{C}{a} \sin\left(\frac{\pi}{a} x\right) \sin(\omega t - \beta z)$$

on the $y = b$ wall the magnetic field points in the positive x-direction, so the current must point in the positive z-direction,

$$J_x = 0 \qquad J_y = 0 \qquad J_z = \frac{C}{a} \sin\left(\frac{\pi}{a} x\right) \sin(\omega t - \beta z)$$

For a given t, z, the currents all flow in the same direction either in the positive or negative z-direction. A good way of illustrating the currents is a contour plot of J_z
8.13

(a) The cutoff frequency is

$$\omega_{c,10} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \sqrt{\left(\frac{\pi}{a}\right)} = 3 \times 10^8 \frac{\pi}{7.62 \times 10^{-2}} = 12.37 \times 10^9\,\mathrm{s}^{-1}$$

The operating frequency is

$$\omega = 1.3 \times \omega_c = 16.08 \times 10^9\,\mathrm{s}^{-1}$$

The intrinsic wave impedance is

$$\eta = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}} = 377 \frac{1}{\sqrt{1 - \left(\frac{1}{1.3}\right)^2}} = 590.0\,\Omega$$

(b) The amplitude of the electric field can be found from equation 8.54,

$$\begin{aligned} P =& \frac{\omega \mu \beta_{10} a^3 b}{(2\pi)^2} H_0^2 \\ =& \frac{\omega \mu \omega \mu a^3 b}{\eta (2\pi)^2} H_0^2 \\ =& \frac{\omega^2 \mu^2 a^3 b}{\mu (2\pi)^2} H_0^2 \end{aligned}$$

and thus

$$H_0 = \sqrt{\frac{P \eta (2\pi)^2}{\omega^2 \mu^2 a^3 b}} = 1.967\,\mathrm{A/m}$$

Next use equation 8.44b, and find that

$$E_0 = \frac{\omega \mu a}{\pi} H_0 = 964.1\,\mathrm{V/m}$$

(c) The expressions for the fields are given in Equations 8.44a-c, and inserting numerical values we get

$$H_z = 1.967 \times \cos\left(\frac{\pi}{0.0762} x\right)$$

$$E_y = -j964.1 \times \sin\left(\frac{\pi}{0.0762} x\right)$$

$$H_x = j1.634 \times \sin\left(\frac{\pi}{0.0762} x\right)$$

$$E_x = 0 \qquad H_y \qquad E_z = 0$$