EE 434 Electricity and Magnetism, Spring 2009
Homework #10 extra credit and exam preparation
Solutions

1. **Derive the equation**

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \omega^2\mu\epsilon + \gamma^2 = 0$$

(which is applied to $\vec{E}$ and $\vec{H}$) for harmonic waves. Begin with the harmonic wave expressions for Faraday's and Ampere's law.

The harmonic forms of Faraday's law and Ampere's law (assuming no volume current) are

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \qquad \nabla \times \vec{H} = j\omega\epsilon\vec{E}$$

Take the curl of Faraday's law,

$$\nabla \times \left(\nabla \times \vec{E}\right) = -j\omega\mu\nabla \times \vec{H}$$
$$\nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2\vec{E} = \omega^2\mu\epsilon\vec{E}$$

Since there are no free charges inside the waveguide we get

$$\nabla^2\vec{E} + \omega^2\mu\epsilon\vec{E} = 0$$
$$\frac{\partial^2\vec{E}}{\partial x^2} + \frac{\partial^2\vec{E}}{\partial y^2} + \frac{\partial^2\vec{E}}{\partial z^2} + \omega^2\mu\epsilon\vec{E} = 0$$

We know the variation of $\vec{E}$ along the z-axis. It is

$$e^{-\gamma z} = e^{-j\beta z}$$

so that we get

$$\frac{\partial^2\vec{E}}{\partial x^2} + \frac{\partial^2\vec{E}}{\partial y^2} + (-j\beta)^2\vec{E} + \omega^2\mu\epsilon\vec{E} = 0$$
$$\frac{\partial^2\vec{E}}{\partial x^2} + \frac{\partial^2\vec{E}}{\partial y^2} - \beta^2\vec{E} + \omega^2\mu\epsilon\vec{E} = 0$$

2. **Explain (using math) why it is sufficient to solve for $\boldsymbol{E_z}$ and $\boldsymbol{H_z}$ in the above equations.**

   We will demonstrate that E_x, E_y, H_x, and H_y can be expressed in terms of E_z and H_z.

3. **Show why $\boldsymbol{E_z}$ is expressed as a product of sines in the waveguide coordinate system we have been using.**

   We begin by noting that the solution for E_z in the equation in problem 1 is

$$E_z(x, y, z, t) = X(x)Y(y)e^{j\omega t - \gamma z}$$

   and that

$$\frac{\partial^2 X}{\partial x^2} = -M^2 X \qquad \frac{\partial^2 Y}{\partial y^2} = -N^2 Y$$

   The solutions to this includes a sum of cosines, sines, and exponentials. The exponential cannot satisfy boundary conditions of zero electric field at the boundary, so we are left with the sum of a cosine and a sine. Since we choose a coordinate system where one boundary in both the X and Y direction is located at zero, and the electric field must vanish at the boundary, only the sine function can survive.

4. **Show why $\boldsymbol{H_z}$ is expressed as a product of cosines.**

   It is a similar argument. H_z can be expressed as the product of two functions, each of which is the sum of a sine and a cosine. The boundary condition for H_z is not easy to work with, so instead we compute E_x and E_y. E_x is the y-derivative of H_z. The cosine in H_z thus corresponds to a sine in E_x. Only the sine variation of E_x as a function of y satisfies the boundary condition, and thus the y-variation of H_z must be according to a cosine. E_y is the x-derivative of H_z. The cosine variation of H_z with x becomes a sine variation of E_y with x. Only the sine varation of E_y with x satisfies the boundary condition, and thus the x-variation of H_z must be according to a cosine.

5. **Explain what the critical frequency is and how it is derived.**

   The critical frequency $\omega_{c,nm}$ is the lowest frequency at which the TE_{mn} and TM_{mn} modes can propagate in a perfectly conducting waveguide without being attenuated.

   It comes from the wave equation for either E_z or H_z,

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \gamma^2 E_z + \omega^2 \mu\epsilon E_z = 0$$

   We found that the first two derivative must be constant and have only discrete values such that

$$-\left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 + \gamma^2 + \omega^2\mu\epsilon = 0$$

from which we can find γ as

$$\gamma = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\mu\epsilon}$$

In order for the electric field to propagate and not be attenuated, γ must be imaginary, so

$$\omega^2\mu\epsilon \geq \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

The minimum frequency at which is this is true is the critical frequency,

$$\omega_c = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

6. **Given the dispersion relation**

$$\boldsymbol{\beta = \omega\sqrt{\mu\epsilon}\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}}$$

show that the group velocity does not exceed the speed of light.

The group velocity is $v_g = \frac{d\omega}{d\beta} = \left(\frac{d\beta}{d\omega}\right)^2$. The second form is easier if we rewrite it as

$$\beta = \sqrt{\mu\epsilon}\sqrt{\omega^2 - \omega_c^2}$$

and then compute

$$\frac{d\beta}{d\omega} = \sqrt{\mu\epsilon}\frac{1}{2\sqrt{\omega^2 - \omega_c^2}}2\omega = \sqrt{\mu\epsilon}\frac{1}{\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}}$$

and then

$$v_g = \left(\frac{d\beta}{d\omega}\right)^{-1} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}$$

we only have a propagating wave for $\omega > \omega_c$, and under those conditions the square root is less than one, and the group velocity is less than $1/\sqrt{\mu\epsilon}$.

7. **In a empty waveguide with $\boldsymbol{a = 2\,\mathrm{cm}}$, $\boldsymbol{b = 3\,\mathrm{cm}}$, the $\mathbf{TE_{11}}$ mode with $\boldsymbol{\omega = 10^{11}\,\mathrm{rad/s}}$, has electric field amplitude $\boldsymbol{E_{x0} = 1\,\mathrm{V/m}}$. Give the amplitude $\boldsymbol{E_{y0}}$, $\boldsymbol{H_{x0}}$, and $\boldsymbol{H_{y0}}$.**

The TE_{11} mode has the axial magnetic field with amplitude A,

$$H_z = A\cos\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{b}y\right)e^{j\omega t-\beta z}$$

From this, the expression for E_x is

$$E_x = -\frac{A\frac{\pi}{a}j\omega\mu}{\gamma^2+\omega^2\mu\epsilon}\sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{b}y\right)e^{j\omega t-\beta z}$$

The absolute value of the factor in front of the harmonic functions is the amplitude, so

$$|E_{x0}| = \frac{A\frac{\pi}{a}\omega\mu}{\gamma^2+\omega^2\mu\epsilon} = 1\,\frac{\mathrm{V}}{\mathrm{m}}$$

For E_y we get

$$|E_{y0}| = \frac{A\frac{\pi}{a}\omega\mu}{\gamma^2+\omega^2\mu\epsilon}$$

So,

$$|E_{y0}| = |E_{x0}|\,\frac{a}{b} = 0.667\,\frac{\mathrm{V}}{\mathrm{m}}$$

For the magnetic field we find that

$$|H_{x0}| = \frac{|\gamma|\,A\frac{\pi}{a}}{\gamma^2+\omega^2\mu\epsilon}$$

we need to find γ, which is

$$\gamma = j\sqrt{\omega^2\mu\epsilon-\left(\frac{\pi}{a}\right)^2-\left(\frac{\pi}{b}\right)^2} = j275.0\,\mathrm{m}^{-1}$$

so we get

$$|H_{x0}| = |E_{x0}|\,\frac{|\gamma|}{\omega\mu} = 2.19\times10^{-3}\,\frac{\mathrm{A}}{\mathrm{m}}$$

and

$$|H_{y0}| = \frac{2}{3}\times2.19\times10^{-3} = 1.46\times10^{-3}\,\frac{\mathrm{A}}{\mathrm{m}}$$

8. **Sketch the magnetic field in the transverse plane in the TM_{32} mode in a rectangular waveguide. Sketch the electric field in the transverse plane in the TE_{32} mode in a rectangular waveguide.**

The TM_{32} mode has

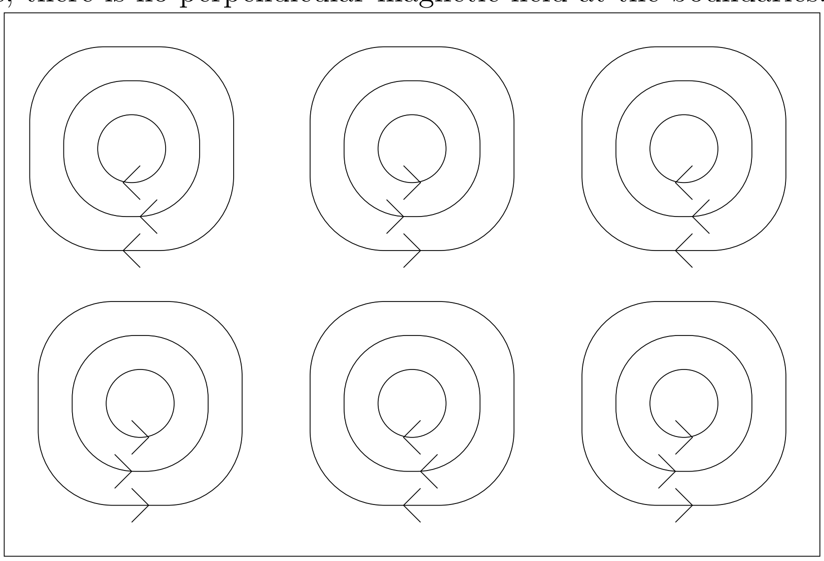
$$E_z = A \sin\left(\frac{3\pi}{a}x\right) \sin\left(\frac{2\pi}{b}y\right)$$

and

$$H_x = A' \sin\left(\frac{3\pi}{a}x\right) \cos\left(\frac{2\pi}{b}y\right)$$

$$H_y = A'' \cos\left(\frac{3\pi}{a}x\right) \sin\left(\frac{2\pi}{b}y\right)$$

Here is the sketch of the magnetic field. Notice that, in agreement with the boundary conditions, there is no perpendicular magnetic field at the boundaries.



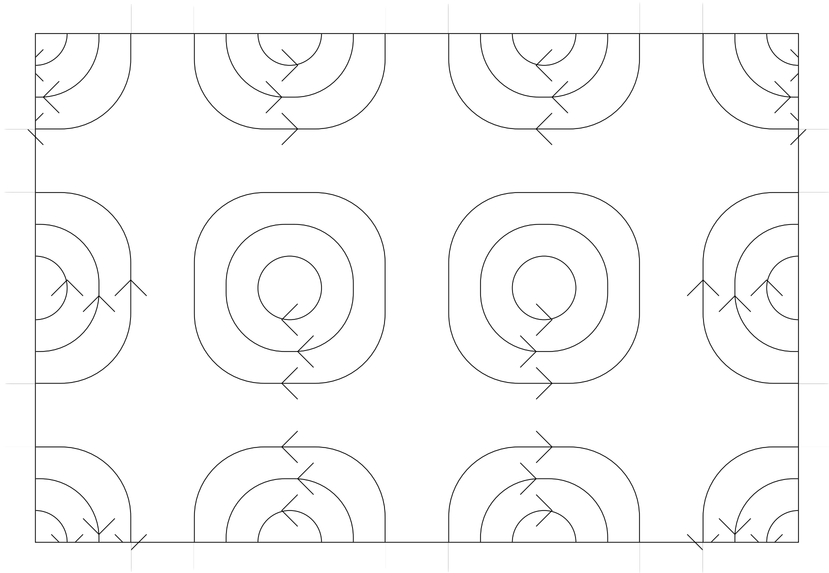
Next we sketch the electric field for a TE_{32} mode. The axial magnetic field is

$$H_z = A \cos\left(\frac{3\pi}{a}x\right) \cos\left(\frac{2\pi}{b}y\right)$$

$$E_x = A' \cos\left(\frac{3\pi}{a}x\right) \sin\left(\frac{2\pi}{b}y\right)$$

$$E_y = A'' \sin\left(\frac{3\pi}{a}x\right) \cos\left(\frac{2\pi}{b}y\right)$$

In this case the field lines begin and end on the waveguide wall, except in the middle where they go around in a loop.



9. **A rectangular waveguide with $a = 2\,\mathrm{cm}$, and $b = 3\,\mathrm{cm}$ is excited with a signal with $\omega = 10^{11}\,\mathrm{rad/s}$. What modes are expected to be excited?**

   The critical frequency for a mn mode is

$$\omega_{c,mn} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

   Inserting for various mn values we get, in $10^9\,\mathrm{rad/s}$,

n / m	0	1	2	3	4
0	0	47.1	94.2	141.3	188.4
1	31.4	56.6	99.3	144.7	191.0
2	62.8	78.5	113.2	154.6	198.6
3	94.2	105.3	133.2	169.8	210.6
4	125.6	134.1	157.0	189.0	226.4

   Valid modes are then TE_{10}, TE_{20}, TE_{01}, TE_{11}, TE_{21}, TE_{02}, TE_{12}, TE_{03}, and TM_{11}, TM_{21}, TM_{12}.

10. **What are the wave impedances for each of the modes in the previous question?**

    For the TE modes we get

$$\eta_{mn} = \sqrt{\frac{\mu}{\epsilon}}\frac{1}{\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}}$$

$$\begin{aligned}
\eta_{10} =& 427.1\,\Omega \\
\eta_{20} =& 1121\,\Omega \\
\eta_{01} =& 396.8\,\Omega \\
\eta_{11} =& 457.0\,\Omega \\
\eta_{21} =& 3141\,\Omega \\
\eta_{02} =& 484.0\,\Omega \\
\eta_{12} =& 608.0\,\Omega \\
\eta_{03} =& 1121.0\,\Omega
\end{aligned}$$

For the TM modes we get

$$\eta_{mn} = \sqrt{\frac{\mu}{\epsilon}}\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}$$

$$\begin{aligned}
\eta_{11} =& 310.6\,\Omega \\
\eta_{21} =& 45.2\,\Omega \\
\eta_{12} =& 233.4\,\Omega
\end{aligned}$$

11. **What are the wavelengths for each of the modes in the previous question?**

For all modes we get, in cm,

$$\lambda_{mn} = \frac{\frac{2\pi}{\omega\sqrt{\mu\epsilon}}}{\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}}$$

$$\begin{aligned}
\lambda_{10} =& 2.14\,\text{cm} \\
\lambda_{20} =& 5.60\,\text{cm} \\
\lambda_{01} =& 1.98\,\text{cm} \\
\lambda_{11} =& 2.28\,\text{cm} \\
\lambda_{21} =& 15.7\,\text{cm} \\
\lambda_{02} =& 2.42\,\text{cm} \\
\lambda_{12} =& 3.04\,\text{cm} \\
\lambda_{03} =& 5.60\,\text{cm}
\end{aligned}$$

12. **What are the group velocities for each of the modes in the previous question?**

In question 6 we derived the expression for the group velocity, which is

$$v_g = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)}$$

$$\begin{aligned}
v_{10} &= 264 \times 10^6\,\mathrm{m/s}\\
v_{20} &= 101 \times 10^6\,\mathrm{m/s}\\
v_{01} &= 285 \times 10^6\,\mathrm{m/s}\\
v_{11} &= 247 \times 10^6\,\mathrm{m/s}\\
v_{21} &= 36.0 \times 10^6\,\mathrm{m/s}\\
v_{02} &= 233 \times 10^6\,\mathrm{m/s}\\
v_{12} &= 186 \times 10^6\,\mathrm{m/s}\\
v_{03} &= 101 \times 10^6\,\mathrm{m/s}
\end{aligned}$$