# EE 434 Electricity and Magnetism, Spring 2009
# Homework #11 Solution

**9.1**

**(a)** For $\theta = \frac{\pi}{2}$, we have $\cos\theta = 0$ and $\sin\theta = 1$. Inserting those into equations 9.14 and 9.15 we get

$$\vec{E} = \frac{j\eta I\,dl}{4\pi\beta}\left(\frac{\beta^2}{r} - \frac{j\beta}{r^2} - \frac{1}{r^3}\right)e^{-j\beta}\hat{\theta}$$

$$\vec{H} = \frac{I\,dl}{4\pi}\left(\frac{j\beta}{r} + \frac{1}{r^2}\right)e^{-j\beta}\hat{\phi}$$

We also have

$$\frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon}} \qquad \beta = 2\pi f\sqrt{\mu\epsilon} = 2\pi \times 10 \times 10^6\sqrt{4\pi \times 10^{-7} \times 8.854 \times 10^{-12}} = 21.0\,\mathrm{cm}$$

and $\eta = 377\,\Omega$, so we get

$$\frac{\eta I\,dl}{4\pi\beta} = \frac{377 \times 1 \times 1}{4\pi \times 0.210} = 143\,\mathrm{V\,m^2}$$

$$\frac{I\,dl}{4\pi} = \frac{1 \times 1}{4\pi} = 0.07958\,\mathrm{A\,m}$$

Now at the three distances we get

| Distance<br>m | $E_\theta$<br>V/m | $H_\phi$<br>A/m |
|---|---|---|
| 1 | 30-136j | 0.0167+0.0796j |
| 5 | 1.2+0.11j | 0.00334+0.00318j |
| 10 | 0.30+0.49j | 0.00167+0.000796j |

**(b)** and **(c)** For this we just compute the far field components (which vary as $1/r$), and the near-field components (which is the rest of the field), and compare them

| Distance | $E_\theta$ | | $H_\phi$ | |
|---|---|---|---|---|
| m | V/m | | A/m | |
| | Near | Far | Near | Far |
| 1 | 30.0-143j | 6.28j | 0.0796 | 0.0167j |
| 5 | 1.20-1.15j | 1.26j | 0.00318 | 0.0033j |
| 10 | 0.300-0.143j | 0.629j | 0.000796 | 0.00167j |

Notice that at $r = 5\,\mathrm{m}$ the near and far field are almost identical, at $r = 1\,\mathrm{m}$ the near field is larger than the far field, and at $r = 10\,\mathrm{m}$ the far field is larger than the near field.

**9.2** The poynting vector is defined as

$$\vec{P} = \vec{E} \times \vec{H}$$

when $\vec{E}$ and $\vec{H}$ are epxressed in their real (sines and cosines) form. If $\vec{E}$ and $\vec{H}$ are expressed as

$$\begin{aligned}\vec{E} =&\vec{E}_0 e^{j\omega t - j\beta}\\ \vec{H} =&\vec{H}_0 e^{j\omega t - j\beta}\end{aligned}$$

then the time-averaged poynting vector is

$$\vec{P}_{\text{av}} = \frac{1}{2} Re\left[\vec{E}_0 \times \vec{H}_0^*\right]$$

In the forms of $\vec{E}$ and $\vec{H}$ in equations 9.14 and 9.15 $\vec{E}_0$ and $\vec{H}_0$ are everything besides the $e^{-j\beta}$ factor. So we write

$$\begin{aligned}\vec{E}_0 =&\frac{j\eta\, I\, dl}{2\pi\beta}\cos\theta\left(\frac{j\beta}{r^2}+\frac{1}{r^3}\right)\hat{r} - \frac{j\eta\, I\, dl}{4\pi\beta}\sin\theta\left(-\frac{\beta^2}{r}+\frac{j\beta}{r^2}+\frac{1}{r^3}\right)\hat{\theta}\\ =&E_{r0}\hat{r}+E_{\theta 0}\hat{\theta}\end{aligned}$$

$$\begin{aligned}\vec{H}_0^* =&\frac{I\, dl}{4\pi}\sin\theta\left(\frac{-j\beta}{r}+\frac{1}{r^2}\right)\hat{\phi}\\ =&H_{\phi 0}\hat{\phi}\end{aligned}$$

The cross-product is then

$$\begin{aligned}\vec{P} =&\frac{1}{2}Re\left[\left(E_{r0}\hat{r}+E_{\theta 0}\hat{\theta}\right)\times H_{\phi 0}\hat{\phi}\right]\\ =&\frac{1}{2}Re\left[-E_{r0}H_{\phi 0}\hat{\theta}+E_{\theta 0}H_{\phi 0}\hat{r}\right]\end{aligned}$$

Let's compute the real component of $E_{r0}H_{\phi 0}$,

$$\begin{aligned}E_{r0}H_{\phi 0} =&\frac{j\eta I^2\, dl^2}{8\pi^2\beta}\left(\frac{-j\beta j\beta}{r^3}+\frac{j\beta}{r^4}+\frac{-j\beta}{r^4}+\frac{1}{r^5}\right)\cos\theta\,\sin\theta\\ =&\frac{j\eta I^2\, dl^2}{8\pi^2\beta}\left(\frac{\beta^2}{r^3}+\frac{1}{r^5}\right)\cos\theta\,\sin\theta\end{aligned}$$

Now notice that the quantity inside the parenthesis is real and there is a $j$ outside, so the whole quantity is imaginary and thus when we take the real component to get the time-averaged Poynting vector, we find that the Poynting vector is zero. Next we look at the radial component of the Poynting vector,

$$\begin{aligned}E_{\theta 0}H_{\phi 0} =&-j\frac{\eta I^2\, dl^2}{16\pi^2\beta}\left(\frac{j\beta^3}{r^2}-\frac{\beta^2}{r^3}+\frac{\beta^2}{r^3}+\frac{j\beta}{r^4}-\frac{j\beta}{r^4}+\frac{1}{r^5}\right)\sin^2\theta\\ =&-j\frac{\eta I^2\, dl^2}{16\pi^2\beta}\left(\frac{j\beta^3}{r^2}+\frac{1}{r^5}\right)\sin^2\theta\\ =&\frac{\eta\, I^2\, dl^2}{16\pi^2\beta}\left(\frac{\beta^3}{r^2}-\frac{j}{r^5}\right)\sin^2\theta\end{aligned}$$

The real component of this is

$$P_{\mathrm{r,av}} = \frac{1}{2}\frac{\eta\, I^2\, dl^2}{16\pi^2\beta}\frac{\beta^3}{r^2} = \frac{\eta\, I^2\, dl^2\beta^2}{32\pi^2 r^2}\sin^2\theta$$

Which is identical to equation 9.21

**9.3** We have a far-field radiation pattern with power

$$\vec{P} = K\frac{\sin\theta\cos\phi}{r^2}\hat{r} \qquad 0 \le \theta \le \pi \qquad\qquad -\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$$

(Probably this means that the radiation field is zero for other values (the "bacward" values) of $\phi$)

**(a)** Half-power beam widths are the angle range around the maximum power point where the power is greater than or equal to half of the peak power. The peak power occurs in the direction $(\theta, \phi) = \left(\frac{\pi}{2}, 0\right)$. The azimuthal beam with is the beam width in the $\phi$ direction. How far do we need to go in $\phi$ before the power drops to half. The answer is

$$\cos\phi = \frac{1}{2} \;\Rightarrow\; \phi = \pm\frac{\pi}{3}$$

The half-power beam widthin the azimuthal direction is therefore,

$$\delta\phi = \frac{\pi}{3} - \left(-\frac{\pi}{3}\right) = \frac{2\pi}{3} = 120^\circ$$

In the $\theta$ direction the maximum is at $\theta = \frac{\pi}{2}$ the half-power occurs at

$$\sin\theta = \frac{1}{2} \;\Rightarrow\; \theta = \frac{\pi}{2} \pm \frac{\pi}{3}$$

The elevation half-power beam width is thus also

$$\delta\theta = 120^\circ$$

**(b)** The first null beam widthe is the range around the beam where the power is non-zero. The angle range to the first null. We can see that in both the azimuthal and elevation planes the first null occurs at ±90° from the peak power, so that the first null beam with is 180° in both planes.

**(c)** Directivity is defined as the power divided by the average power per unit area (average over all directions). If the total power emitted by the antenna is $P_{\mathrm{tot}}$, then the average power per unit area is

$$P_{avg} = \frac{P_{\mathrm{tot}}}{4\pi r^2}$$

and the directive gain is

$$D(\theta, \phi) = \frac{P\,(r, \theta, \phi)}{P_{\mathrm{avg}}\,(r, \theta, \phi)}$$

and the directivity, $D_0$, is simply the maximum value of $D(\theta, \phi)$. Now, we just need to compute the integral of $P$ over a sphere... or is it over half a sphere, since the power was only defined over half the sphere in the problem. I am not sure what the author has in mind here, but the difference is a factor of two (i.e. divide by $4\pi r^2$ versus by $2\pi r^2$) so I will just consider half the sphere and hope for the best. So,

$$\begin{aligned} P_{\text{tot}} &= \int_0^{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} K \frac{\sin\theta \cos\phi}{r^2} r^2 \sin\theta \, d\phi \, d\theta \\ &= \int_0^{\pi} K \frac{2\sin^2\theta}{d} \theta \\ &= \pi K \end{aligned}$$

The average power is then

$$P_{\text{avg}} = \frac{\pi K}{2\pi r^2} = \frac{K}{2r^2}$$

and the directive gain is

$$D(\theta, \phi) = K \frac{\sin\theta \, \cos\phi}{r^2} \frac{2r^2}{K} = 2\sin\theta \, \cos\phi$$

and the directivity is the maximum of $D$,

$$D_0 = 2$$