Partial Fraction Decomposition

Partial fractions: (take a fraction and write this fraction as a sum of two or more similar fractions.)

Example:

$$\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$$

(The LCD is 6.)

$$\frac{1}{2}\left(\frac{3}{3}\right)+\frac{1}{3}\left(\frac{2}{2}\right)=\frac{5}{6}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$$

Simplified, $\frac{5}{6}$ can be simplified and written as two separate fractions: $\frac{1}{2} \,\&\, \frac{1}{3}$.

Steps of Decomposition of Partial Fractions.

1. Divide if improper: If $N(x)/D(x)$ is an improper fraction, divide the denominator into the numerator to obtain:

$$\frac{N(x)}{D(x)}=(polynomial)+\frac{N_1(x)}{D(x)}$$

and apply Steps 2, 3, and 4 (below) to the proper rational expression $N_1(x)/D(x)$.

2. Factor denominator: Completely factor the denominator into factors of the form

$$(px+q)^m \text{ and } \left(ax^2+bx+c\right)^n$$

where $\left(ax^2+bx+c\right)^n$ is irreducible.

3. Linear factors: For each factor of the form $(px+q)^m$, the partial fraction decomposition must include the following sum of *m* fractions.

$$\frac{A_1}{(px+q)}+\frac{A_2}{(px+q)^2}+\cdots\frac{A_n}{(px+q)^n}$$

4. Quadratic factors: For each factor of the form $(ax^2+bx+c)^n$, the partial fraction decomposition must include the following sum and *n* fractions.

$$\frac{B_1x+C_1}{ax^2+bx+c}+\frac{B_2x+C_2}{(ax^2+bx+c)^2}+\cdots\frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

Example:

$$\frac{x+7}{x^2-x-6}$$

SOLUTION: Write partial fraction decomposition.

Step 1: Factor denominator.

$$\frac{x+7}{x^2-x-6}=\frac{}{(x+2)(x-3)}=\frac{A}{(x+2)}+\frac{B}{(x-3)}$$

Step 2: Now find A and B by multiplying both sides by the LCM. In this case the LCM is $(x+2)and(x-3)$.

$$(x+2)(x-3)\left[\frac{x+7}{(x+2)(x-3)}\right]=(x+2)(x-3)$$

$$\cancel{(x+2)}\,\cancel{(x-3)}\left[\frac{x+7}{\cancel{(x+2)}\,\cancel{(x-3)}}\right]=(x+2)(x-3)\left[\frac{A}{(x+2)}+\frac{B}{(x-3)}\right]$$

$$x+7=\frac{(x+2)(x-3)A}{(x+2)}+\frac{(x+2)(x-3)B}{(x-3)}$$

$$x+7=A(x-3)+B(x+2)$$

$$x+7=Ax-3A+Bx+2B$$

Step 3: Equate undetermined coefficients.

$1=A+B$ (first equation)

Step 4: Equate coefficients of constant terms:

$7=-3A+2B$ (second equation)

Step 5: Solve algebraically with method “2 equations and 2 unknowns”. This is done by substitution, graphing, or the addition method.

$$\begin{array}{c} +3 \\ \hline 1 \end{array}\left\{\begin{array}{l} A+\ \ B=1 \\ 3A+2B=7 \end{array}\right\}$$

$$\begin{array}{r} 3A+3B=3 \\ \underline{-3A+2B=7} \\ 0+5B=10 \\ B=2 \end{array}$$

Now substitute equation 2 into equation 1:

$$A + B = 1$$
$$A + 2 = 1 - 2$$
$$A = -1$$

Thus, the solution is:

$$\left[\frac{x+7}{(x+2)(x-3)}\right] = \left[\frac{A}{(x+2)} + \frac{B}{(x-3)}\right] = \left[\frac{x+7}{(x+2)(x-3)}\right] = \left[\frac{-1}{(x+2)} + \frac{2}{(x-3)}\right]$$

Now check:

$$\left[\frac{-1}{(x+2)} + \frac{2}{(x-3)}\right]$$

LCM = $(x+2)$ and $(x-3)$

$$\frac{-1}{x+2}\frac{(x-3)}{(x-3)} + \frac{2}{(x-3)}\frac{(x+2)}{(x+2)}$$

$$\frac{-x+3+2x+4}{(x-3)(x+2)} = \frac{(x-7)}{x^2-x-6}$$

$\dfrac{(x+7)}{x^2-x-6}$ is the original problem!