Solutions to August 28-30 discussion problems

Problem 1.1

$$\Delta p A = \rho g h A + \left(\frac{\dot{h}}{2}\right) D\left(2\pi L\right) + \left(\frac{\ddot{h}}{2}\right)\left(\rho L A\right)$$

(a) Laplace transform the differential equation. I will use $h(s)$ to indicate the laplace transform of $h(t)$, and $\Delta p(s)$ to indicate the Laplace transform of $\Delta p(t)$.

$$\Delta p(s)A = \rho g h(s)A + sh(s)D\pi L + s^2 h(s)\frac{\rho L A}{2}$$

Rearrange to get

$$\Delta p(s)A = h(s)\left(\rho g A + sD\pi L + s^2\frac{\rho L A}{2}\right)$$

and

$$\frac{h(s)}{\Delta p(s)} = \frac{A}{\rho g A + sD\pi L + s^2\frac{\rho L A}{2}}$$

To put it in time-constant form we make the constant term 1,

$$\begin{aligned}\frac{h(s)}{\Delta p(s)} &= \frac{\frac{A}{\rho g A}}{1 + s\frac{D\pi L}{\rho g A} + s^2\frac{\rho L A}{2\rho g A}} \\ &= \frac{\frac{1}{\rho g}}{1 + s\frac{D\pi L}{\rho g A} + s^2\frac{L}{2g}}\end{aligned}$$

(b) There are several ways of doing this. One is to compare the differential equation with differential equation 1.4a in Northrop. Another is find the roots of the quadratic and compare those to the roots of the polynomial form of Equation 1.4a, which are

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

(Note that the book has a mistake in its expression for s, in the paragraph immediately following Equations 1.4a-c.)

I will take the second approach, finding the roots of the denominator,

$$
\begin{aligned}
s &= -\frac{\pi D}{\rho A} \pm \sqrt{\left(\frac{\pi D}{\rho A}\right)^2 - \frac{2g}{L}} \\
&= -\frac{\pi D}{\rho A} \pm j\sqrt{\frac{2g}{L} - \left(\frac{\pi D}{\rho A}\right)^2}
\end{aligned}
$$

which gives us

$$
\zeta\omega_n = \frac{\pi D}{\rho A} \qquad \omega_n\sqrt{1-\zeta^2} = \sqrt{\frac{2g}{L} - \left(\frac{\pi D}{\rho A}\right)^2}
$$

which we can solve as

$$
\omega_n = \sqrt{\frac{2g}{L}} \qquad \zeta = \frac{\pi D}{\rho A}\sqrt{\frac{L}{2g}}
$$

Next I will take the first approach, and re-write the differential equation in the form of 1.4a.

$$
\ddot{h}\frac{\rho L A}{2} = -\dot{h}D\pi L - h\rho g A + \Delta p A
$$

$$
\begin{aligned}
\ddot{h} &= -\dot{h}\frac{2D\pi L}{\rho L A} - h\frac{2\rho g A}{\rho L A} + \Delta p\frac{2A}{\rho L A} \\
&= -\dot{h}\frac{2D\pi}{\rho A} - h\frac{2g}{L} + \Delta p\frac{2}{\rho L}
\end{aligned}
$$

Then

$$
\omega_n^2 = \frac{2g}{L} \Rightarrow \omega_n = \sqrt{\frac{2g}{L}}
$$

and

$$
2\zeta\omega_n = \frac{2D\pi}{\rho A}
$$

$$
\zeta = \frac{D\pi}{\rho A\omega_n} = \frac{D\pi}{\rho A}\sqrt{\frac{L}{2g}}
$$

(c)

$$\omega_n = \sqrt{2g}L = \sqrt{2 * 9.82}1 = 4.43\text{s}^{-1}$$

$$D = \sqrt{\frac{2\zeta^2\rho^2 A^2 g}{\pi^2 L}} = 1.77 \times 10^{-3}\frac{\text{kg}}{\text{m}\cdot\text{s}}$$

Problem 1.2

$$\frac{V_o(s)}{\ddot{X}(s)} = \frac{K_A}{(\tau s+1)^2}$$

A unit step of acceleration means that $\ddot{x} = u(t)$, and $\ddot{X} = \frac{1}{s}$, so that

$$V_o(s) = \frac{K_A}{s(\tau s+1)^2} = \frac{\frac{K_A}{\tau^2}}{s(s+\frac{1}{\tau})^2} \tag{1}$$

I choose the last form because the Laplace tables I have access to have a factor 1 in front of s. We need to inverse Laplace transform this expression to get the time-evolution in order to be able to answer questions A, B, and C. To inverse transform it we first expand the fraction. It can be shown that

$$V_o(s) = \frac{A}{s} + \frac{B}{s+\frac{1}{\tau}} + \frac{C}{\left(s+\frac{1}{\tau}\right)^2} \tag{2}$$

Next, we need to find A, B, and C. We do this by re-combining the fractions,

$$\begin{aligned} V_o(s) =& \frac{A\left(s+\frac{1}{\tau}\right)^2 + Bs\left(s+\frac{1}{\tau}\right) + Cs}{s\left(s+\frac{1}{\tau}\right)^2} \\ =& \frac{As^2 + \frac{2sA}{\tau} + \frac{A}{\tau^2} + Bs^2 + \frac{Bs}{\tau} + Cs}{s\left(s+\frac{1}{\tau}\right)^2} \end{aligned}$$

Now we collect the terms with the same power of s,

$$V_o(s) = \frac{s^2\left(A+B\right) + s\left(\frac{2A}{\tau} + \frac{B}{\tau} + C\right) + \frac{A}{\tau^2}}{s\left(s+\frac{1}{\tau}\right)^2}$$

Next compare the numerator in this expression with the numerator in Equation 1,

$$\begin{aligned} s^0 :& \frac{A}{\tau^2} = \frac{K_A}{\tau^2} \Rightarrow A = K_A \\ s^2 :& A + B = 0 \Rightarrow B = -A = -K_A \\ s^1 :& \frac{2A}{\tau} + \frac{B}{\tau} + C = 0 \Rightarrow C = -\frac{B}{\tau} - \frac{2A}{\tau} = \frac{K_A}{\tau} - \frac{2K_A}{\tau} = \frac{-K_A}{\tau} \end{aligned}$$

Inserting A, B, and C into Equation 2 we get

$$V_o(s) = \frac{K_A}{s} - \frac{K_A}{s + \frac{1}{\tau}} + \frac{\frac{K_A}{\tau}}{\left(s + \frac{1}{\tau}\right)^2}$$

We can inverse Laplace transform this expression to get

$$v_o(t) = K_A \left(1 - e^{\frac{t}{\tau}} - \frac{t}{\tau} e^{\frac{t}{\tau}}\right)$$

We can solve this expression numerically to obtain the answers

(A) $t = 4.744\tau = 4.744s$

(B) $t = 6.639\tau = 6.639s$

(C) $t = 9.234\tau = 9.234s$