Solutions to Homework #2

Taylor 3.42
(A)

$$a = g \sin(\theta)$$

$g = 9.81\frac{\mathrm{m}}{\mathrm{s}^2}$, $\theta = 5.4° = 0.09425$, so

$$a = 9.81 \times \sin(5.4°)\,\frac{\mathrm{m}}{\mathrm{s}^2} = 0.9232\frac{\mathrm{m}}{\mathrm{s}^2}$$

(many digits included because precision is still unknown)

$$\begin{aligned}\delta a^2 &= \left(\frac{\partial a}{\partial \theta}\delta\theta\right)^2 \\ &= \left(g\cos(\theta)\,\delta\theta\right)^2\end{aligned}$$

so

$$\delta a = g\cos(\theta)\,\delta\theta$$

Additionally $\delta\theta = 0.1°$, so

$$\delta a = 9.81 \times \cos(5.4°) \times \left[0.1° \times \frac{\pi}{180°}\right]\frac{\mathrm{m}}{\mathrm{s}^2} = 0.0170\frac{\mathrm{m}}{\mathrm{s}^2}$$

The final answer is then

$$a = 0.923 \pm 0.017\frac{\mathrm{m}}{\mathrm{s}^2} \quad \text{or} \quad a = 0.92 \pm 0.02\frac{\mathrm{m}}{\mathrm{s}^2}$$

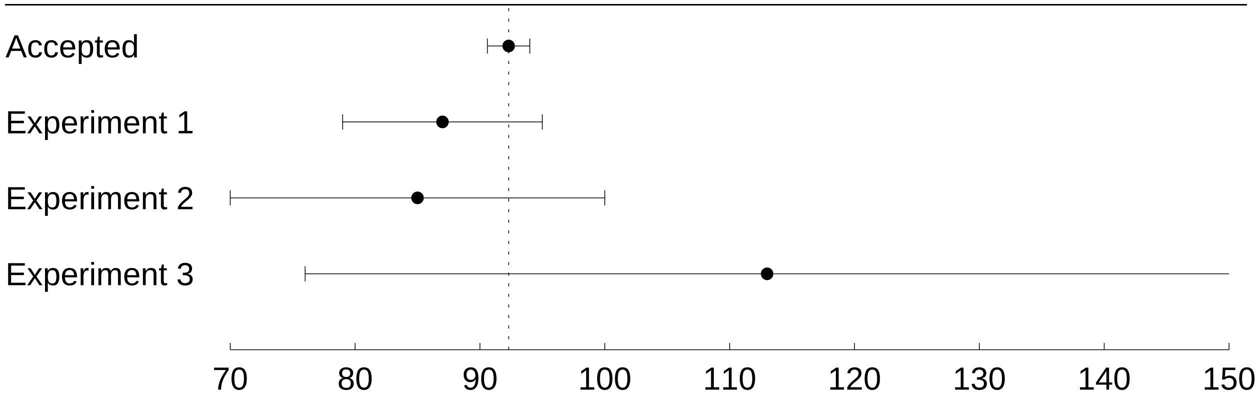
Tip of the day: How many digits should I keep in intermediate results?
You can think of the rounding that you do when cutting off digits in an intermediate result as a source of error. This error will be plus or minus one in the last digit that you keep. This error will propagate through future calculations exactly like the measurement errors. You want the error in your final result to be dominated by the uncertainty in the measurements. To achieve that you make your roundoff errors much smaller than the measurement error. How many digits do you keep then? Well it depends on how many expressions your values are propagating through. If there is a long sequence of consecutive calculations and you compute an intermediate result between each one, you need to be careful to include enough digits so that roundoff errors do not propagate. Better safe than sorry.
(B)

We can calculate the acceleration via formula (3.33),

$$a = \frac{l^2}{2s}\left(\frac{1}{t_2^2} - \frac{1}{t_1^2}\right)$$

We can compute the uncertainty in the acceleration by the general formula involving parial derivatives, or we can follow the procedure listed in section 3.10. Let's follow the procedure in section 3.10.

t_1(s)	t_2(s)	$\frac{1}{t_1^2}$	$\frac{1}{t_2^2}$	$\frac{1}{t_2^2} - \frac{1}{t_1^2}$	a
all ± 0.001	all ± 0.001				(cm/s^2)
$0.054 \pm 2\%$	$0.031 \pm 3\%$	343 ± 14	1040 ± 62	698 ± 64	87 ± 8
$0.038 \pm 2.6\%$	$0.027 \pm 4\%$	693 ± 36	1372 ± 102	679 ± 108	85 ± 14
$0.025 \pm 4\%$	$0.020 \pm 5\%$	1600 ± 128	2500 ± 250	900 ± 281	113 ± 35



All of the uncertainty intervals for the experiments overlap with each other and with the uncertainty interval for the accepted value. The series of experiments are therefore consistent with a constant value for a.

Taylor 3.48

We wish to perform error calculation on

$$q = \frac{x+y}{x+z}$$

If we do this as a stepwise calculation we first compute the uncertainties on

$$a = x + y \;\text{ and }\; b = x + z$$

as

$$\delta a = \delta x \;\text{ and }\; \delta b = \delta x$$

and then the uncertainty on q as

$$\begin{aligned}
\delta q^2 =& q^2 \left(\left(\frac{\delta a}{a}\right)^2 + \left(\frac{\delta b}{b}\right)^2 \right) \\
=& \left(\frac{x+y}{x+z}\right)^2 \left(\left(\frac{\delta x}{x+y}\right)^2 + \left(\frac{\delta x}{x+z}\right)^2 \right) \\
=& \frac{\delta x^2}{(x+z)^2} + \frac{(x+y)^2\,\delta x^2}{(x+z)^4} \\
=& \delta x^2 \frac{\left[(x+z)^2 + (x+y)^2\right]}{(x+z)^4}
\end{aligned}$$

so that

$$\delta q = \delta x \frac{\sqrt{(x+z)^2 + (x+y)^2}}{(x+z)^2}$$

With the generalized formula 3.47 we compute δq^2 directly as

$$\delta q^2 = \left(\frac{\partial q}{\partial x}\delta x\right)^2 = \left(\frac{x+z-(x+y)}{(x+z)^2}\delta x\right)^2 = \left(\frac{z-y}{(x+z)^2}\delta x\right)^2 = \frac{(z-y)^2\,\delta x^2}{(x+z)^4}$$

so that

$$\delta q = \frac{|z-y|}{(x+z)^2}\delta x$$

(A)
$x = 20 \pm 1$, $y = 2$, $z = 0$

$$q = \frac{x+y}{x+z} = \frac{20+2}{20+0} = 1.1000.....$$

1. Stepwise calculation

   $$\delta q = 1 \times \frac{\sqrt{(20)^2 + (20+2)^2}}{(20)^2} = 0.074$$

   so that $q = 1.10 \pm 0.07$

2. Differential calculation

   $$\delta q = \frac{|0-2|}{(20+0)^2} \times 1 = 0.005$$

   so that $q = 1.100 \pm 0.005$

The two methods compute uncertainties which are different by more than a factor of 10!!
(B)
$x = 20 \pm 1$, $y = -40$, $z = 0$

$$q = \frac{x+y}{x+z} = \frac{20-40}{20+0} = -1$$

1. Stepwise calculation

   $$\delta q = 1 \times \frac{\sqrt{(20+0)^2 + (20-40)^2}}{(20+0)^2} = 0.071$$

   so that $q = -1.00 \pm 0.07$

2. Differential calculation

$$\delta q = \frac{|0-(-40)|}{(20+0)^2} \times 1 = 0.1$$

so that $q = -1.0 \pm 0.1$

In case (A) the stepwise rule overestimates the uncertainty on q. In ase (B) the stepwise rule underestimates the uncertainty on q. The stepwise rule cannot be trusted to give an accurate uncertainty when an uncertain quantity appears both in the numerator and in the denominator of a fraction.

Taylor 3.49

(A)

$$f = \frac{pq}{p+q}$$

$$\begin{aligned}\delta f^2 &= \left(\frac{\partial f}{\partial p}\delta p\right)^2 + \left(\frac{\partial f}{\partial q}\delta q\right)^2 \\ &= \left(\frac{q(p+q)-pq}{(p+q)^2}\delta p\right)^2 + \left(\frac{p(p+q)-pq}{(p+q)^2}\delta q\right)^2 \\ &= \frac{q^4\delta p^2 + p^4\delta q^2}{(p+q)^4}\end{aligned}$$

so that

$$\delta f = \frac{\sqrt{q^4\delta p^2 + p^4\delta q^2}}{(p+q)^2}$$

(B)

$$f = \frac{pq}{p+q} = \frac{pq\frac{1}{pq}}{\frac{p}{pq}+\frac{q}{pq}} = \frac{1}{\frac{1}{q}+\frac{1}{p}}$$

Stepwise computation. Let

$$a = \frac{1}{p} \qquad b = \frac{1}{q} \qquad c = \frac{1}{p} + \frac{1}{q} = a + b$$

Then

$$\frac{\delta a}{a} = \frac{\delta p}{p}$$

so that

$$\delta a = a\frac{\delta p}{p} = \frac{1}{p}\frac{\delta p}{p} = \frac{\delta p}{p^2}$$

Similarly

$$\delta b = \frac{\delta q}{q^2}$$

Then

$$\delta c = \sqrt{\delta a^2 + \delta b^2} = \sqrt{\frac{\delta p^2}{p^4} + \frac{\delta q^2}{q^4}}$$

Finally

$$\begin{aligned}
\delta f =& f\frac{\delta c}{c} = \frac{pq}{p+q}\frac{\sqrt{\frac{\delta p^2}{p^4} + \frac{\delta q^2}{q^4}}}{\frac{1}{p} + \frac{1}{q}} = \frac{pq}{p+q}\frac{pq\sqrt{\frac{\delta p^2}{p^4} + \frac{\delta q^2}{q^4}}}{pq\left(\frac{1}{p} + \frac{1}{q}\right)} \\
=& p^2q^2\frac{\sqrt{\frac{\delta p^2}{p^4} + \frac{\delta q^2}{q^4}}}{(p+q)^2} = \frac{\sqrt{\frac{p^4q^4\delta p^2}{p^4} + \frac{p^4q^4\delta q^2}{q^4}}}{(p+q)^2} \\
=& \frac{\sqrt{q^4\delta p^2 + p^4\delta q^2}}{(p+q)^2}
\end{aligned}$$

Taylor 3.50

$$q = \frac{x+2}{x + y\cos(4\theta)}$$

with $x = 10 \pm 2$, $y = 7 \pm 1$, and $\theta = 40 \pm 3°$
compute δq:

$$\begin{aligned}
\delta q^2 &= \left(\frac{\partial q}{\partial x}\delta x\right)^2 + \left(\frac{\partial q}{\partial y}\delta y\right)^2 + \left(\frac{\partial q}{\partial \theta}\delta \theta\right)^2 \\
&= \left(\frac{y\cos(4\theta) - 2}{(x + y\cos(4\theta))^2}\delta x\right)^2 + \left(\frac{-(x+2)\cos(4\theta)}{(x + y\cos(4\theta))^2}\delta y\right)^2 + \left(\frac{-4(x+2)\,y\sin(4\theta)}{(x + y\cos(4\theta))^2}\delta \theta\right)^2 \\
&= \frac{(y\cos(4\theta) - 2)^2\,\delta x^2 + (x+2)^2\cos^2(4\theta)\,\delta y^2 + 16\,(x+2)^2\,y^2\sin^2(4\theta)\,\delta\theta^2}{(x + y\cos(4\theta))^4}
\end{aligned}$$

so that

$$\delta q = \frac{\sqrt{(y\cos(4\theta) - 2)^2\,\delta x^2 + (x+2)^2\cos^2(4\theta)\,\delta y^2 + 16\,(x+2)^2\,y^2\sin^2(4\theta)\,\delta\theta^2}}{(x + y\cos(4\theta))^2}$$

Inserting (and being careful to convert degrees to radians), we get

$$q = \frac{10+2}{10 + 7 \times \cos\left(4 \times \frac{40}{180}\pi\right)} = 3.50657...$$

$$\delta q = 1.82$$

The final value is then

$$q = 3.5 \pm 1.8$$

Derive Northrop Equation 6.58

We start by writing an equation for current balance at the location marked (0) in Figure 6.25. That point is also a virtual ground. Total current flowing out of point (0) is zero,

$$0 = i_p(t) - \frac{V_0}{R_F} - C_F\frac{dV_0}{dt}$$

Take the laplace transform

$$\begin{aligned} I_p(s) =& \frac{V_0(s)}{R_F} + C_F V_0'(s) \\ =& \frac{V_0(s)}{R_F} + sC_F V_0(s) \end{aligned}$$

We are given (Equation 6.49) that

$$\frac{I_P(s)}{P_i} = \frac{sK_P A\Theta}{s\Theta C_T + 1}$$

which we can insert so we get

$$P_i\frac{sK_P A\Theta}{s\Theta C_T + 1} = \frac{V_0(s)}{R_F} + sC_F V_0(s)$$

$$P_i\frac{sK_P A\Theta}{s\Theta C_T + 1} = V_0(s)\left[\frac{1}{R_F} + sC_F\right]$$

$$\begin{aligned} V_0(s) =& P_i\frac{sK_P A\Theta}{[s\Theta C_T + 1]\left[\frac{1}{R_F} + sC_F\right]} \\ =& P_i\frac{\frac{sK_P A\Theta}{\Theta C_T C_F}}{\left[s + \frac{1}{\Theta C_T}\right]\left[\frac{1}{R_F C_F} + s\right]} \\ =& P_i\frac{sK_P A/C_T C_F}{\left[s + \frac{1}{\Theta C_T}\right]\left[\frac{1}{R_F C_F} + s\right]} \end{aligned}$$

For a step function in radiation intensity at time $t = 0$ with final intensity P_{io}, for which the Laplace transform is

$$P_i(s) = \frac{P_{io}}{s}$$

the expression for $V_0(s)$ becomes

$$V_0(s) = P_{io} \frac{K_P A / C_T C_F}{\left[s + \frac{1}{\Theta C_T}\right]\left[\frac{1}{R_F C_F} + s\right]}$$

Northrop 6.2

(A)

The dark current is (using sign convention on Figure P6.2)

$$I_D = \frac{-V_{\text{bias}}}{R_D} = \frac{-V_{\text{bias}}}{\frac{\rho L}{A}} = \frac{-V_{\text{bias}} A}{\rho L} = \frac{+10 \times 0.001 \times 0.05}{2.3 \times 10^3 \times 0.5} = 0.435\,\mu\text{A}$$

The current across the resistor is I_D, so we have

$$V_{0(\text{dark})} = I_D \times 10^6 \Omega = 0.435\,\text{V}$$

(B)

The photo current will equal the dark current when the photo resistance equals the dark resistance, so

$$R_D = \frac{\rho L}{A} = R_P = \frac{L^2}{q\eta\tau_p\left(\mu_p + \mu_n\right)} \frac{hc}{P_i \lambda}$$

or

$$\begin{aligned}
P_i =& \frac{L^2}{q\eta\tau_p\left(\mu_p + \mu_n\right)} \frac{hc}{\lambda} \frac{A}{\rho L} \\
=& \frac{0.005^2}{1.60 \times 10^{-19} \times 0.8 \times 10^{-4} \times (0.045 + 0.15)} \times \frac{6.624 \times 10^{-34} \times 3 \times 10^8}{6.33 \times 10^{-7}} \\
& \times \frac{0.5 \times 0.05}{2.3 \times 10^3 \times 0.5} \\
=& 1.3671 \times 10^{-7}\,\text{W} = 0.13671\,\mu\text{W}
\end{aligned}$$

This corresponds to a radiant intensity of

$$\frac{P_i}{A} = \frac{1.3671 \times 10^{-7}}{0.05 \times 0.5} = 5.47\,\mu\text{W/cm}^2$$

(C)

This is explained in Figure 6.6

Northrop 6.3

(A)

The Laplace transform of the response function of a first-order system is

$$\frac{V_o(s)}{V_i(s)} = \frac{K}{s+\frac{1}{\tau}}$$

where τ is the response time-constant of the system. If we compare this expression to Equation 6.46 we find that

$$\frac{\Delta T(s)}{P_i(s)} = \frac{\Theta}{s\Theta C_T + 1} = \frac{\frac{1}{C_T}}{s+\frac{1}{\Theta C_T}}$$

Thus,

$$\tau = \Theta C_T = \Theta c A h = 200 \times 2.34 \times 10^6 \times \pi \times \left(0.5 \times 10^{-2}\right)^2 \times 25.46 \times 10^{-6} = 0.936\,\mathrm{s}$$

(B)
The power to current response function is

$$\frac{I_P(s)}{P_i(s)} = \frac{sK_P A\Theta}{s\Theta C_T + 1}$$

The Laplace transform of the power is

$$P_i = \Phi_i(s) = \Phi_0 \frac{\left(1-e^{-sT}\right)}{s}$$

so that the Laplace transform of the current is

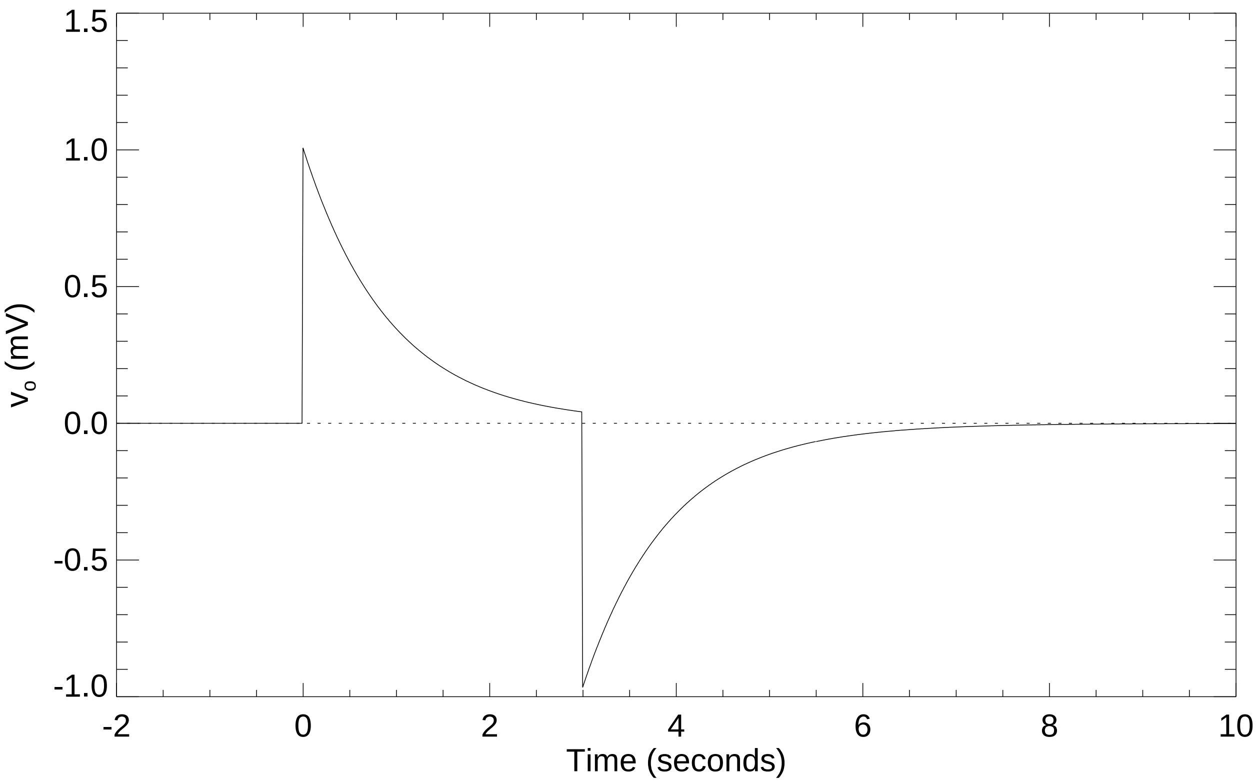
$$\begin{aligned} I_p(s) =& \frac{\Phi_0\left(1-e^{-sT}\right)K_P A\Theta}{s\Theta C_T + 1} \\ =& \frac{\Phi_0 K_P A}{C_T}\left[\frac{1}{s+\frac{1}{\Theta C_T}} - \frac{e^{-sT}}{s+\frac{1}{\Theta C_T}}\right] \end{aligned}$$

We can inverse Laplace transform to

$$i_p(t) = \frac{\Phi_0 K_P A}{C_T}\left[\exp\left(-\frac{t}{\Theta C_T}\right)u\left(t\right) - \exp\left(-\frac{t-T}{\Theta C_T}\right)u\left(t-T\right)\right]$$

If we name the $10^{10}\Omega$ resistor in Figure P6.3 R, then we can write

$$v_o(t) = i_p(t)R = \frac{R\Phi_0 K_P A}{C_T}\left[\exp\left(-\frac{t}{\Theta C_T}\right)u\left(t\right) - \exp\left(-\frac{t-T}{\Theta C_T}\right)u\left(t-T\right)\right]$$


1.5
1.0
0.5
0.0
-0.5
-1.0
v_o (mV)
-2
0
2
4
6
8
10
Time (seconds)