Solutions to Homework #3

Northrop 6.1
(A) Find expression for V_o in terms of R, ΔR, and V_B
The voltage at the positive terminal.

$$V_X = \frac{V_B R}{R + \Delta R + R} = V_B \frac{R}{2R + \Delta R}$$

This is also the voltage on the negative terminal. The node equation at the negative terminal is.

$$\frac{V_o - V_X}{R + \Delta R} + \frac{V_B - V_X}{R} = 0$$

Isolate V_x

$$\frac{V_o}{R + \Delta R} - \frac{V_X}{R + \Delta R} + \frac{V_B}{R} - \frac{V_X}{R} = 0$$

$$\frac{V_X}{R + \Delta R} + \frac{V_X}{R} = \frac{V_o}{R + \Delta R} + \frac{V_B}{R}$$

$$\begin{aligned} V_X &= \left[\frac{V_o}{R + \Delta R} + \frac{V_B}{R}\right] \frac{1}{\frac{1}{R+\Delta R} + \frac{1}{R}} \\ &= \left[\frac{V_o}{R + \Delta R} + \frac{V_B}{R}\right] \frac{(R + \Delta R) R}{R + R + \Delta R} \\ &= \left[\frac{V_o}{R + \Delta R} + \frac{V_B}{R}\right] \frac{(R + \Delta R) R}{2R + \Delta R} \end{aligned}$$

Eliminate V_X:

$$V_B \frac{R}{2R + \Delta R} = \left[\frac{V_o}{R + \Delta R} + \frac{V_B}{R}\right] \frac{(R + \Delta R) R}{2R + \Delta R}$$

$$V_B R = \left[\frac{V_o}{R + \Delta R} + \frac{V_B}{R}\right] (R + \Delta R) R$$

$$V_B R = V_o R + V_B (R + \Delta R)$$

$$V_B R = V_o R + V_B R + V_B \Delta R$$

$$V_o R = -V_B \Delta R$$

$$V_o = -\frac{\Delta R}{R} V_B$$

(B) Compute the strain, ϵ
We are given the relationship

$$\Delta R = R_{27}\left[\epsilon \mathrm{GF} + \alpha\left(T - 27\right)\right],$$

Isolate ϵ,

$$\frac{\Delta R}{R_{27}} = \epsilon \mathrm{GF} + \alpha\left(T - 27\right)$$

$$\epsilon = \frac{1}{\mathrm{GF}}\left[\frac{\Delta R}{R_{27}} - \alpha\left(T - 27\right)\right]$$

From the answer to question (A) we can obtain an expression for ΔR,

$$\Delta R = -\frac{V_o R}{V_B}$$

Insert this, noting that $R = R_{27}$

$$\begin{aligned}\epsilon =& \frac{1}{\mathrm{GF}}\left[-\frac{V_o}{V_B} - \alpha\left(T - 27\right)\right]\\ =& \frac{1}{2.9}\left[\frac{35.4 \times 10^{-3}}{5} - 1 \times 10^{-4}\left(67 - 27\right)\right]\\ =& 1.06 \times 10^{-3}\end{aligned}$$

The wires are stretched 0.38%.
(C) In this case the wires are not stretched (The tank is at atmospheric pressure and at a lower temperature than 27°C, so the tank circumference must be smaller than the unstretched wires). So $\epsilon = 0$.

This time, isolate T in the given relationship (setting $\epsilon = 0$)

$$\frac{\Delta R}{R_{27}} = \alpha\left(T - 27\right)$$

$$T = \frac{1}{\alpha}\frac{\Delta R}{R_{27}} + 27$$

Insert the expression for ΔR noting again that $R = R_{27}$

$$\begin{aligned}T =& -\frac{1}{\alpha}\frac{V_o}{V_B} + 27\\ =& -\frac{1}{1 \times 10^{-4}}\frac{10 \times 10^{-3}}{5} + 27\\ =& 7^\circ\mathrm{C}\end{aligned}$$

Northrop 6.21
(A) Give expressions for $v_{Hz}(t)$ and $v_{Hz}(t)$

I will call the angular frequency of the variation ω_0. In that case $v_{Hy}(t)$ is re-written as

$$v_{Hy}(t) = K_H\,|B|\sin(\omega_0 t)\sin(\beta)$$

Similarly, by inspecting Figure P6.21A we can see that

$$v_{Hx}(t) = K_H\,|B|\sin(\omega_0 t)\cos(\beta)\cos(\theta)$$

and

$$v_{Hz}(t) = K_H\,|B|\sin(\omega_0 t)\cos(\beta)\sin(\theta)$$

(B) Derive expressions for $v_2(t)$, $v_3(t)$, and $V_4(s)$.
We see immediately that $v_3(t) = v_2(t)^2$
Let's write the node equation for the negative input of the leftmost IOA in the circuit.

$$-\frac{v_{Hz}^2}{10R_1} - \frac{v_{Hx}^2}{10R_1} - \frac{v_{Hy}}{10R_1} + \frac{v_3}{R_2} = 0$$

$$-\frac{v_{Hz}^2 + v_{Hx}^2 + v_{Hy}^2}{10R_1} + \frac{v_2^2}{R_2} = 0$$

$$\begin{aligned}
v_2(t) =& \sqrt{\frac{R_2}{10R_1}\left(v_{Hz}^2 + v_{Hx}^2 + v_{Hy}^2\right)} \\
=& \sqrt{\frac{R_2}{10R_1}K_H^2\,|B|^2\sin^2(\omega_0 t)\left[\cos^2(\beta)\sin^2(\theta) + \cos^2(\beta)\cos^2(\theta) + \sin^2(\beta)\right]} \\
=& \sqrt{\frac{R_2}{10R_1}}K_H\,|B|\sin(\omega_0 t)
\end{aligned}$$

Since $v_3 = v_2^2$

$$\begin{aligned}
v_3(t) =& \frac{R_2}{10R_1}K_H^2 B^2\sin^2(\omega_0 t) \\
=& \frac{R_2}{10R_1}K_H^2 B^2\frac{1}{2}\left[1 - \cos(2\omega_0 t)\right]
\end{aligned}$$

Let's compute the transfer function of the second amplifier stage. That is the transfer function from V_3 to V_A, with V_A being the output of the second op-amp. Let's write the node equation for the negative input of the second op-amp.

$$\frac{V_A}{\frac{1}{sC}} + \frac{V_A}{10R} + \frac{V_3}{R} = 0$$

$$sCV_A + \frac{V_A}{10R} + \frac{V_3}{R} = 0$$

$$V_A\left(sC + \frac{1}{10R}\right) = -\frac{V_3}{R}$$

$$\begin{aligned}\frac{V_A}{V_3} &= -\frac{\frac{1}{R}}{sC+\frac{1}{10R}} \\ &= -\frac{10}{10sCR+1}\end{aligned}$$

Next, let's write the node equation for the negative terminal of the third op-amp,
Let's Laplace transform $v_3(t)$, assuming that the signal begins at $t = 0$

$$V_3(s) = \frac{K_H^2 B^2}{20}\frac{R_2}{R_1}\left[\frac{1}{s} - \frac{s}{s^2+4\omega_o^2}\right]$$

Then V_A is

$$V_A(s) = V_3(s)\frac{V_A}{V_3} = -\frac{10}{10sCR+1}\frac{K_H^2 B^2}{20}\frac{R_2}{R_1}\left[\frac{1}{s} - \frac{s}{s^2+4\omega_o^2}\right]$$

Now for the final op-amp we have

$$\frac{V_4^2}{R} + \frac{V_A}{R} = 0 \Longrightarrow V_4 = \sqrt{-V_A}$$

$$\begin{aligned}V_4 =& \sqrt{\frac{10}{10sCR+1}\frac{K_H^2 B^2}{20}\frac{R_2}{R_1}\left[\frac{1}{s} - \frac{s}{s^2+4\omega_o^2}\right]} \\ =& \frac{B}{\sqrt{2}}K_H\sqrt{\frac{R_2}{R_1}}\sqrt{\frac{1}{10sCR+1}\left[\frac{1}{s} - \frac{s}{s^2+4\omega_o^2}\right]} \\ =& \text{RMS}\,(B)\,K_H\sqrt{\frac{R_2}{R_1}}\sqrt{\frac{1}{10sCR+1}\left[\frac{1}{s} - \frac{s}{s^2+4\omega_o^2}\right]}\end{aligned}$$

Northrop 2.9
The gain is the absolute value of the transfer function. More precisely, it is the absolute value of the ratio $\frac{V_o(s)}{V_1(s)}$ or $\frac{V_o(s)}{I_1(s)}$.
(A)
There is no current flowing in the conductor G_1. The node equation at the negative terminal of the IOA is

$$I_1 + \frac{V_o}{R_F} = 0$$

so

$$\frac{V_o}{I_1} = -R_F$$

The gain is then $G = -R_F$ (In Ohms).
(B)

There is no current flowing in R_1. The voltage at the positive terminal is

$$V_+ = V_o \frac{R_F}{R_L + R_F}$$

The problem is that this circuit is not a negative feedback circuit, and is not stable. Imagine a small positive noise-spike on V_o. That will result in a small positive noise spike at the positive input terminal. That in turn will result in a large positive spike on V_o, which will drive V_+ even higher, and so on in a runaway feedback loop. The Gain is effectively infinite.

If on the other hand we switched the negative and positive terminals on the IOA, a positive spike on V_o would result in a positive spike on V_- (voltage on the negative terminal), which would result in a negative correction on V_o. If the positive and negative terminal are switched, we can write

$$V_- = V_o \frac{R_F}{R_L + R_F}$$

so that

$$G = \frac{V_o}{V_-} = \frac{R_L + R_F}{R_L} = 1 + \frac{R_F}{R_L}$$

However this is not the answer to the stated problem.
(C)
No current is flowing in the conductor G_1. Write the node equation at the negative input terminal.

$$I_1 + C_F \frac{dV_o}{dt} = 0$$

Laplace transform

$$I_1(s) + sC_F V_o(s) = 0$$

$$\frac{V_o(s)}{I_1(s)} = \frac{1}{sC_F}$$

The gain is

$$G = \left| \frac{V_o(j\omega)}{I_(j\omega)} \right| = \left| \frac{1}{j\omega C_F} \right| = \frac{1}{\omega C_F}$$

Notice that the gain is infinite for a DC current.
(D)
Current flowing through C_2 towards negative IOA terminal.

$$I_2(t) = C_2 \frac{dV_o}{dt} \Longrightarrow I_2(s) = sC_2 V_o(s)$$

Current flowing through R_2 towards negative IOA terminal

$$I_3(t) = \frac{V_o}{R_2} \Longrightarrow I_3(s) = \frac{V_o(s)}{R_2}$$

Current flowing through C_1 and R_1 towards negative IOA terminal. Let V_X be the voltage between C_1 and R_1

$$I_1(t) = C_1 \frac{dV_1 - V_X}{dt} \qquad\qquad V_X = I_1 R_1$$

$$I_1(t) = C_1 \frac{dV_1}{dt} - C_1 R_1 \frac{dI_1}{dt} \Longrightarrow I_1(s) = sC_1 V_1(s) - sC_1 R_1 I_1(s)$$

Re-arrange to isolate I_1

$$I_1(s)\left[1 + sC_1 R_1\right] = sC_1 V_1(s) \Longrightarrow I_1(s) = V_1(s) \frac{sC_1}{1 + sC_1 R_1}$$

Next write the node equation at the negative terminal of the IOA

$$I_1(s) + I_2(s) + I_3(s) = 0$$

$$V_1(s) \frac{sC_1}{1 + sC_1 R_1} + sC_2 V_o(s) + \frac{V_o(s)}{R_2} = 0$$

$$V_o(s)\left[sC_2 + \frac{1}{R_2}\right] = -\frac{sC_1}{1 + sC_1 R_1} V_1(s)$$

$$\begin{aligned}
\frac{V_o(s)}{V_1(s)} &= -\frac{\frac{sC_1}{1+sC_1R_1}}{sC_2 + \frac{1}{R_2}} \\
&= -\frac{sC_1}{\left(sC_2 + \frac{1}{R_2}\right)(1 + sC_1 R_1)} \\
&= -\frac{sC_1 R_2}{(sC_2 R_2 + 1)(sC_1 R_1 + 1)}
\end{aligned}$$

Insert $s = j\omega$

$$\frac{V_o(s)}{V_1(s)} = -\frac{j\omega C_1 R_2}{(j\omega C_2 R_2 + 1)(j\omega C_1 R_1 + 1)}$$

The Gain is then

$$G = \left|\frac{V_o(s)}{V_1(s)}\right| = \frac{C_1 R_2}{\sqrt{(\omega^2 C_2^2 R_2^2 + 1)(\omega^2 C_1^2 R_1^2 + 1)}}$$

(E)

The node equation at the negative terminal

$$\frac{V_1}{R_1} + \frac{V_o}{R_2 + \frac{1}{sC_2}} = 0$$

$$\frac{V_1}{R_1} = -\frac{V_o}{R_2 + \frac{1}{sC_2}}$$

$$\begin{aligned}\frac{V_o}{V_1} &= -\frac{R_2 + \frac{1}{sC_2}}{R_1} \\ &= -\left[\frac{R_2}{R_1} + \frac{1}{sR_1C_2}\right]\end{aligned}$$

Insert $s = j\omega$

$$\frac{V_o}{V_1} = -\left[\frac{R_2}{R_1} - j\frac{1}{\omega^2 R_1 C_2}\right]$$

The gain is

$$\begin{aligned}G = \left|\frac{V_o}{V_1}\right| &= \sqrt{\frac{R_2^2}{R_1^2} + \frac{1}{\omega^2 R_1^2 C_2^2}} \\ &= \sqrt{\frac{\omega^2 R_1^2 R_2^2 C_2^2 + R_1^2}{\omega^2 R_1^4 C_2^2}} \\ &= \frac{\sqrt{\omega^2 R_2^2 C_2^2 + 1}}{\omega R_1 C_2}\end{aligned}$$

(F)
Node equation at negative terminal of IOA

$$I_{R_1} + I_{C_1} + I_{C_2} = 0$$

The currents are

$$I_{R_1} = \frac{V_1}{R_1} \qquad I_{C_1} = \frac{V_1}{\frac{1}{sC_1}} = sC_1V_1 \qquad I_{C_2} = \frac{V_o}{\frac{1}{sC_2}} = sC_2V_o$$

Insert

$$\frac{V_1}{R_1} + sC_1V_1 + sC_2V_o = 0$$

$$\begin{aligned}\frac{V_o}{V_1} &= -\frac{\frac{1}{R_1} + sC_1}{sC_2} \\ &= -\left[\frac{1}{sR_1C_2} + \frac{C_1}{C_2}\right]\end{aligned}$$

Insert $s = j\omega$

$$\frac{V_o}{V_1} = -\left[\frac{C_1}{C_2} - j\frac{1}{\omega R_1 C_2}\right]$$

The gain is

$$\begin{aligned} G = \left|\frac{V_o}{V_1}\right| &= \sqrt{\frac{C_1^2}{C_2^2} + \frac{1}{\omega^2 R_1^2 C_2^2}} \\ &= \sqrt{\frac{\omega^2 R_1^2 C_1^2 C_2^2 + C_2^2}{\omega^2 R_1^2 C_2^4}} \\ &= \frac{\sqrt{\omega^2 R_1^2 C_1^2 + 1}}{\omega R_1 C_2} \end{aligned}$$

(G)
The node equation at the negative terminal of the IOA

$$\frac{V_1}{R_1 + \frac{1}{sC_1}} + \frac{V_o}{R_2} = 0$$

$$\frac{V_o}{V_1} = -\frac{R_2}{R_1 + \frac{1}{sC_1}}$$

Insert $s = j\omega$

$$\begin{aligned} \frac{V_o}{V_1} &= -\frac{R_2}{R_1 + \frac{1}{j\omega C_1}} \\ &= -\frac{j\omega C_1 R_2}{j\omega C_1 R_1 + 1} \end{aligned}$$

The gain is

$$G = \left|\frac{V_o}{V_1}\right| = \frac{\omega C_1 R_2}{\sqrt{\omega^2 C_1^2 R_1^2 + 1}}$$

Northrop 2.11
The common mode and differential mode inputs are

$$V_{1d} = V_{1c} = \frac{V_s}{2}$$

We can write the equations

$$V_o = A_D V_{1d} + A_C V_{1c} = (A_D + A_C)\frac{V_s}{2} \qquad \text{CMRR} = 120\,\text{dB} = 10^6 = \frac{A_D}{A_C}$$

In terms of RMS

$$\mathrm{RMS}(V_o) = (A_D + A_C)\,\frac{1}{2}\mathrm{RMS}(V_s)$$

First find A_D

$$A_D = 10^6 A_C \qquad\qquad A_C = \frac{\mathrm{RMS}(V_o)}{\frac{1}{2}\mathrm{RMS}(V_s)} - A_D$$

$$A_D = 10^6\left[\frac{\mathrm{RMS}(V_o)}{\frac{1}{2}\mathrm{RMS}(V_s)} - A_D\right]$$

$$A_D = \frac{10^6}{10^6+1}\frac{\mathrm{RMS}(V_o)}{\frac{1}{2}\mathrm{RMS}(V_s)}$$

With $\mathrm{RMS}(V_o) = 1.0$ and $\mathrm{RMS}(V_s) = \frac{0.004}{\sqrt{2}}$, we get

$$A_D = 707$$

Then

$$A_C = \frac{A_D}{10^6} = 707 \times 10^{-6} = 7.07 \times 10^{-4}$$

Norhtrop 2.12
(A)

$$\mathrm{CMRR} = \frac{10^3}{10^{-2}} = 10^5 = 20\log_{10} 10^5\,\mathrm{dB} = 100\,\mathrm{dB}$$

(B)

$$V_o = A_D V_{1d} + A_C V_{1c}$$

$$V_1(t) = A\sin(\omega_0 t) \qquad\qquad V_1'(t) = -A\sin(\omega_0 t) + B\sin(\omega_1 t)$$

$$V_{1d} = \frac{V_1 - V_1'}{2} = A\sin(\omega_0 t) - \frac{B}{2}\sin(\omega_1 t)$$

$$V_{1c} = \frac{V_1 + V_1'}{2} = \frac{B}{2}\sin(\omega_1 t)$$

$$\begin{aligned} V_o =& A_D A\sin(\omega_0 t) - A_D\frac{B}{2}\sin(\omega_1 t) + A_C\frac{B}{2}\sin(\omega_1 t) \\ =& A_D A\sin(\omega_0 t) + (A_C - A_D)\frac{B}{2}\sin(\omega_1 t) \end{aligned}$$

Inserting $A_D = 10^3$, $A_C = 10^{-2}$, $A = 0.002$, $B = 2$, $\omega_0 = 2\pi 400$, and $\omega_1 = 377$, we get

$$V_o = 2\sin(2\pi 400 t) - 999.99\sin(377 t)$$