Solutions to Homework #4

Northrop 2.10
Node equation for V_2

$$\frac{V_B}{R_3} + \frac{V_1}{R + \Delta R} = 0$$

Node equation for V_3

$$\frac{V_B}{R_3} + \frac{V_1}{R} + \frac{V_o}{R_2} = 0$$

Isolate V_1 in each equation and eliminate it

$$-V_B \frac{R + \Delta R}{R_3} = V_1 = -V_B \frac{R}{R_3} - V_o \frac{R}{R_2}$$

Now isloate V_o

$$V_o \frac{R}{R_2} = V_B \frac{R + \Delta R}{R_3} - V_B \frac{R}{R_3}$$

$$\begin{aligned} V_o =& V_B \frac{R_2}{R} \left(\frac{R + \Delta R}{R_3} - \frac{R}{R_3} \right) \\ =& V_B \frac{R_2}{R} \frac{R + \Delta R - R}{R_3} \\ =& V_B \frac{R_2 \Delta R}{R_3 R} \end{aligned}$$

Northrop 3.1
(A)
The RMS noise voltage is $\sqrt{\langle v^2 \rangle}$. Over a frequency band $[f_1; f_2]$, the RMS noise voltage is

$$\text{RMS}_{\text{noise}} = \sqrt{\langle v^2 \rangle} = \sqrt{\int_{f_1}^{f_2} S df}$$

For a resistor the noise power spectrum is

$$S_R(f) = 4kTR$$

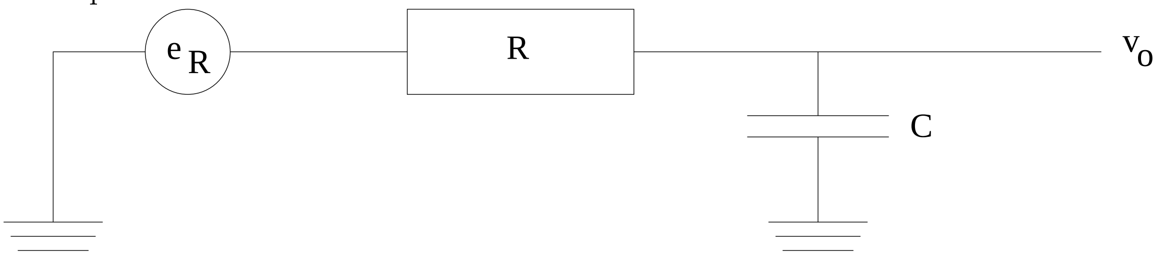
So that

$$\sqrt{\langle v_R^2 \rangle} = \sqrt{\int_{f_1}^{f_2} 4kTR df} = \sqrt{4kTR(f_2 - f_1)}$$

For $R = 1\,\text{M}\Omega$, $4kT = 1.656 \times 10^{-20}$, $f_1 = 100\,\text{Hz}$, and $f_2 = 20\,\text{kHz}$, we get

$$\sqrt{\langle v_R^2 \rangle} = \sqrt{1.656 \times 10^{-20} \times 1 \times 10^6 \times (20 \times 10^3 - 100)} = 18\,\mu\text{V}$$

(B)
The equivalent circuit looks like this



Now the resistor and capacitor form a RC low-pass filter with transfer function

$$H(s) = \frac{1}{sRC + 1}$$

Then the output noise power spectrum, $S_{no}(f)$ at the point v_o can (according to Northrop Equation 3.35) be expressed as

$$S_{no}(f)) = S_R \left|H(2\pi j f)\right|^2$$

Inserting $S_R = 4kTR$ we get

$$\begin{aligned} S_{no}(f) =& 4kTR \left|\frac{1}{j2\pi fRC + 1}\right|^2 \\ =& 4kTR \frac{1}{(1 + j2\pi fRC)(1 - j2\pi fRC)} \\ =& \frac{4kTR}{1 + (2\pi fRC)^2} \end{aligned}$$

(C)
We are being asked to compute the RMS voltage at the point v_o, which, according to Northrop Equation 3.14 is

$$
\begin{aligned}
\sqrt{\langle v_o^2\rangle} =& \sqrt{\int_0^\infty S_{no}(f)df} \\
=& \sqrt{\int_0^\infty \frac{4kTR}{1+(2\pi fRC)^2}df} \\
=& \sqrt{\int_o^\infty \frac{\frac{4kTR}{(2\pi RC)^2}}{\frac{1}{(2\pi RC)^2}+f}df} \\
=& \sqrt{\frac{4kTR}{2\pi RC}}\sqrt{\int_0^\infty \frac{\frac{1}{2\pi RC}}{\sqrt{\frac{1}{(2\pi RC)^2}+f}}df} \\
=& \sqrt{\frac{4kTR}{2\pi RC}}\sqrt{\frac{\pi}{2}} \\
=& \sqrt{\frac{kT}{C}}
\end{aligned}
$$

Insert given values to get

$$
\sqrt{\langle v_o^2\rangle} = \sqrt{\frac{\frac{1}{4}\times 1.656\times 10^{-20}}{23\times 10^{-12}}} = 1.3\times 10^{-5}\,\mathrm{V}
$$

Northrop 3.2
(A)
The input signal to the amplifier is

$$
V_i = v_s(t) = V_s \sin(2\pi f_0 t)
$$

The output signal is also a sinusoid,

$$
V_o = V_s K_V \left|\frac{1}{\tau j 2\pi f_0 + 1}\right| \sin(2\pi f_0 t + \theta)
$$

where θ is used to indicate the phase shift introduced by the low-pass filter. The RMS of V_o is then

$$
\begin{aligned}
\sqrt{\langle V_o^2\rangle} =& \frac{1}{\sqrt{2}} V_s K_V \left|\frac{1}{\tau j 2\pi f_0 + 1}\right| \\
=& \frac{V_s K_V}{\sqrt{2}} \frac{1}{\sqrt{(1+\tau j 2\pi f_0)(1-\tau j 2\pi f_0)}} \\
=& \frac{V_s K_V}{\sqrt{2}} \frac{1}{\sqrt{1+(\tau 2\pi f_0)^2}}
\end{aligned}
$$

If we plug in the numbers given, $V_s = 10 \times 10^{-6}\,\mu\text{V}$, $K_V = 10^3$, $\tau = 0.01\,\text{s}$, $f_0 = 100\,\text{Hz}$, we get

$$\sqrt{\langle V_o^2 \rangle} = \frac{10 \times 10^{-6} \times 10^3}{\sqrt{2}} \frac{1}{\sqrt{1 + (0.01 \times 2\pi \times 100)^2}} = 0.011\,\text{V}$$

(B)
The RMS of the noise voltage at the point V_o is

$$\sqrt{\langle V_{oN}^2 \rangle} = \sqrt{\int_o^\infty S_o(f) df}$$

where $S_o(f)$ is the noise power spectrum. The noise power spectrum at the point V_o is related to the noise power spectrum at the point V_o' by the equation

$$S_o(f) = S_o'(f) \left| H \left(j2\pi f \right) \right|^2 = S_o'(f) \left| \frac{1}{\tau j 2\pi f + 1} \right|^2$$

The noise power spectrum at the point V_o' is related to the power spectrum at the point V_i by

$$S_o'(f) = S_i(f) K_V^2$$

The noise power spectrum at the point V_i is

$$S_i(f) = 4kTR + e_{na}^2$$

We can insert these expressions to obtain the RMS noise voltage at the point V_o,

$$\begin{aligned} \sqrt{\langle V_{oN}^2 \rangle} &= \sqrt{\int_0^\infty \left(4kTR + e_{na}^2\right) K_V^2 \left| \frac{1}{\tau j 2\pi f + 1} \right|^2 df} \\ &= \sqrt{4kTR + e_{na}^2} \sqrt{\int_0^\infty \left| \frac{K_V}{\tau j 2\pi f + 1} \right|^2 df} \end{aligned}$$

The integral can be found in the first line of Table 3.1 on page 125. The entire expression thus reduces to

$$\begin{aligned} \sqrt{\langle V_{oN}^2 \rangle} &= \sqrt{4kTR + e_{na}^2} \sqrt{2\pi \frac{K_V^2}{4 \times 2\pi\tau}} \\ &= \sqrt{\left(4kTR + e_{na}^2\right) \frac{K_V^2}{4\tau}} \end{aligned}$$

Insert given values to get

$$\sqrt{\langle V_{oN}^2 \rangle} = 74\,\mu\text{V}$$

(C)
The SNR is

$$
\begin{aligned}
\mathrm{SNR}_o =& \frac{\langle V_o^2 \rangle}{\langle V_{oN}^2 \rangle} = \frac{\frac{V_s^2 K_V^2}{2} \frac{1}{1+(\tau 2\pi f_0)^2}}{\left(4kTR + e_{na}^2\right) \frac{K_V^2}{4\tau}} \\
=& \frac{V_s^2 K_V^2}{2\left(4kTR + e_{an}^2\right) \frac{K_V^2}{4}} \frac{\tau}{1+(\tau 2\pi f_0)^2} \\
=& \frac{V_s^2 K_V^2}{2\left(4kTR + e_{an}^2\right) \frac{K_V^2}{4}} \frac{1}{\frac{1}{\tau} + \tau\left(2\pi f_0\right)^2}
\end{aligned}
$$

In order to maximize SNR_o we must minimize the denominator of the second fraction. The minimum is the solution to

$$
\begin{aligned}
0 =& \frac{d}{d\tau}\left(\frac{1}{\tau} + \tau\left(2\pi f_0\right)^2\right) \\
=& -\frac{1}{\tau^2} + \left(2\pi f_0\right)^2
\end{aligned}
$$

$$
\tau = \frac{1}{2\pi f_0}
$$

Inserting given values we get

$$
\tau = \frac{1}{2\pi 100} = 1.6 \times 10^{-3}
$$

Insert this value to obtain the maximum SNR_o,

$$
\mathrm{SNR}_o = 7.2
$$

Northrop 3.3
(A)
We first need to compute the output noise. Each resistor produces noise voltage with a power spectrum

$$
S_R = 4kTR
$$

Because the bridge is a 50/50 voltage divider with respect to each noise voltage source, only half of the noise amplitude leaves the bridge, from each resistor. This is a quarter of the power spectrum. So the power spectrum at the output node of the bridge is

$$
S_{\mathrm{bridge}}(f) = 4 \times \frac{4kTR}{2^2} = 4kTR
$$

The noise power spectrum at the input to the differential amplifier is

$$
S_i(f) = 4kTR + e_{na}^2
$$

The noise at the point $V_o'$is

$$S_o'(f) = \left(4kTR + e_{na}^2\right) K_D^2$$

The MS output at the point V_o is

$$\langle v_o^2 \rangle = \int_0^\infty S_o'(f) \left|H\left(j2\pi f\right)\right|^2 df = S_o'B = \left(4kTR + e_{na}^2\right) K_D^2 B$$

Next, let's find the mean-squared variation of the signal. The input to the negative node of the differential amplifier is

$$V_i'(t) = V_B(t) \frac{R - \Delta R}{R + \Delta R + R - \Delta R} = V_B(t) \frac{R - \Delta R}{2R}$$

The input to the positive node of the differential amplifier is

$$V_i(t) = V_B(t) \frac{R + \Delta R}{R + \Delta R + R - \Delta R} = V_B(t) \frac{R + \Delta R}{2R}$$

The output of the differential amplifier, V_o' is then

$$V_o'(t) = \left(V_i(t) - V_i'(t)\right) K_D = V_B(t) \left(\frac{R + \Delta R}{2R} - \frac{R - \Delta R}{2R}\right) K_D = V_B(t) \frac{\Delta R}{R} K_D$$

The signal passes through the bandpass filter with unit amplification. The mean-squared value of the output signal voltage, $V_o(t)$ is therefore $\langle V_o^2 \rangle$ is therefore

$$\langle V_o^2 \rangle = \langle V_o'^2 \rangle = \langle V_B(t)^2 \rangle \left(\frac{\Delta R}{R}\right)^2 K_D^2 = \frac{V_B}{2} \left(\frac{\Delta R}{R}\right)^2 K_D^2$$

We now want to find the value of ΔR which satisifes the equation

$$\langle v_o^2 \rangle = \langle V_o^2 \rangle$$

$$\left(4kTR + e_{na}^2\right) K_D^2 B = \frac{V_B^2}{2} \left(\frac{\Delta R}{R}\right)^2 K_D^2$$

$$\Delta R = \sqrt{\frac{2R^2}{V_B^2} \left(4kTR + e_{na}^2\right) B}$$

Now use $R = 330\,\Omega$, $V_B = 7.071\,\mathrm{V}$, $T = 300\,\mathrm{K}$, $e_{na} = 15\,\frac{\mathrm{nV}}{\sqrt{\mathrm{Hz}}}$, and $B = 10\,Hz$.

$$\begin{aligned}\Delta R =& \sqrt{\frac{2 \times 330^2}{7.071^2} \left(4 \times 1.38 \times 10^{-23} \times 300 \times 330 + [15 \times 10^{-9}]^2\right) \times 10} \\ =& 1.0 \times 10^{-6}\,\Omega\end{aligned}$$

Northrop 3.9

(A)
Node equation at the point where the four wires meet

$$\frac{V_o}{R_F} + I_D - I_B = 0$$

where I_D is the diode current. In the dark, $I_D = I_{DK}$ and we get

$$\begin{aligned} V_o =& R_F \left(I_B - I_{DK}\right) \\ =& 10^6 \left(35 \times 10^{-15} + 4 \times 10^{-9}\right) \\ =& 4\,\text{mV} \end{aligned}$$

(If we assume that I_B flows in to each amplifier input terminal then the answer is also $V_o = 4\,\text{mV}$)
(B)
Let's follow the example in Northrop section 3.8.5. In that example we must replace R_1 with R_D in the positions where it is used as an amplitification factor, and replace the noise from R_1 with the noise from the diode. The resistance of the diode is

$$R_D = R_{DK} || R_P$$

where $R_{DK} = \frac{V_S}{I_{DK}}$ and $R_P = \frac{V_S}{I_P}$. The output noise is

$$\langle v_o^2 \rangle = \left[S_D \left(\frac{R_F}{R_D}\right)^2 + \left(i_{na}^2 + \frac{4kT}{R_F}\right) R_F^2 + e_{na}^2 \left(1 + \frac{R_F}{R_D}\right)^2 \right] B$$

The output voltage from the signal is

$$V_o = R_F \left(I_B - I_D\right)$$

so that the mean-squared output voltage is

$$\langle V_o^2 \rangle = R_F^2 \left(I_B - I_D\right)^2 = R_F^2 \left(I_B - I_{DK} - KP_i\right)^2$$

According to Northrop section 3.8.5 the noise power spectrum at the output is

$$S_o = S_D \left(\frac{R_F}{R_D}\right)^2 + \left(i_{na}^2 + \frac{4kT}{R_F}\right) R_F^2 + e_{na}^2 \left(1 + \frac{R_F}{R_D}\right)^2$$

where

$$\begin{aligned} & S_D = i_{sh}^2 R_D^2 = 2qI_D R_D^2 \qquad I_D = I_{DK} + I_P \qquad I_P = KP_i \\ & \frac{1}{R_D} = \frac{1}{R_{DK}} + \frac{1}{R_P} = \frac{I_{DK}}{V_s} + \frac{KP_i}{V_s} \end{aligned}$$

so that

$$
\begin{aligned}
S_o =& R_F^2 \left[\frac{S_D}{R_D^2} + i_{na}^2 + \frac{4kT}{R_F} + e_{na}^2 \left(\frac{1}{R_F} + \frac{1}{R_D} \right)^2 \right] \\
=& R_F^2 \left[2qI_D + i_{na}^2 + \frac{4kT}{R_F} + e_{na}^2 \left(\frac{1}{R_F} + \frac{I_{DK}}{V_s} + \frac{KP_i}{V_s} \right)^2 \right] \\
=& R_F^2 \left[2q\left(I_{DK} + KP_i\right) + i_{na}^2 + \frac{4kT}{R_F} + e_{na}^2 \left(\frac{1}{R_F} + \frac{I_{DK} + KP_i}{V_s} \right)^2 \right]
\end{aligned}
$$

The means-squared noise at the output is

$$
\langle v_o^2 \rangle = S_o B
$$

the output SNR is

$$
\begin{aligned}
\mathrm{SNR}_o =& \frac{R_F^2 \left(I_B - I_{DK} - KP_i\right)^2}{R_F^2 \left[2q\left(I_{DK} + KP_i\right) + i_{na}^2 + \frac{4kT}{R_F} + e_{na}^2 \left(\frac{1}{R_F} + \frac{I_{DK}+KP_i}{V_s} \right)^2 \right] B} \\
=& \frac{\left(I_B - I_{DK} - KP_i\right)^2}{\left[2q\left(I_{DK} + KP_i\right) + i_{na}^2 + \frac{4kT}{R_F} + e_{na}^2 \left(\frac{1}{R_F} + \frac{I_{DK}+KP_i}{V_s} \right)^2 \right] B}
\end{aligned}
$$

(C)

I think there is a mistake in the assignment. I think the equation in this qustion should read $V_o = V_{oD} + \Delta V_o$. We wish to find the situation in which the MS voltage due to the photo current only is equal to the MS voltage due to the noise currents only. We can do that by setting $\mathrm{SNR}_o = 1$, and setting $I_{DK} = 0$ in the numerator, and setting $P_i = 0$ in the denominator.

$$
1 = \frac{\left(I_B - KP_i\right)^2}{\left[2qI_{DK} + i_{na}^2 + \frac{4kT}{R_F} + e_{na}^2 \left(\frac{1}{R_F} + \frac{I_{DK}}{V_s} \right)^2 \right] B}
$$

$$
\left[2qI_{DK} + i_{na}^2 + \frac{4kT}{R_F} + e_{na}^2 \left(\frac{1}{R_F} + \frac{I_{DK}}{V_s} \right)^2 \right] B = \left(I_B - KP_i\right)^2
$$

$$
P_i = \frac{I_B - \sqrt{\left[2qI_{DK} + i_{na}^2 + \frac{4kT}{R_F} + e_{na}^2 \left(\frac{1}{R_F} + \frac{I_{DK}}{V_s} \right)^2 \right] B}}{K}
$$

Using the quantities given in the problem we can compute

$$
P_i = 1.17 \times 10^{-8}\mathrm{W}
$$