Solutions to Homework #5

Expand argument in first half of Northrop section 3.10.2.3

We start with Faraday's law,

$$\int_l \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_A \vec{B} \cdot d\vec{A}$$

The path integral proceeds in the right-hand positive direction with respect to the direction of $\vec{A}$ around the edge of the area A. If we are simply interested in the voltage induced in one loop due to the current flowing in another, we can write

$$V_B = -M_{AB} \frac{dI_A}{dt}$$

We can laplace transform the above expression and insert $s = j\omega$,

$$V_B(s) = -M_{AB} j\omega I_A(s)$$

where the polarity of the current and voltages must be considered correctly, and can be found from the vector expression. The "dot" convention for transformers is such that when the time derivative of the current flowing into the dotted terminal of coil A, the voltage at the dotted terminal on coil B is positive, with the mutual inductance M_{AB} being the constant of proportionality. When I_A and V_B are defined according to the dot convention, then is written

$$V_B(t) = M \frac{dI_A(t)}{dt}$$

or

$$V_B(s) = M j\omega I_A(s)$$

Since the positive direction of all currents agrees with the dot convention, we can write the voltage in coil 2 due to the current in coil 1 as

$$V_{21}(s) = M_{12} j\omega I_1(s)$$

Because coil 2 and coil s are wound together, a positive current in coil s must result in a negative voltage in coil 2,

$$V_{2s}(s) = -M_{s2} j\omega I_s(s)$$

so that

$$V_2(s) = V_{21}(s) + V_{2s}(s) = M_{12} j\omega I_1(s) - M_{s2} j\omega I_s(s)$$

Next we make compute the current in the shield loop,

$$I_s(s) = \frac{V_s(s)}{j\omega L_s + R_s}$$

where

$$V_s(s) = j\omega M_{1s} I_1(s) - j\omega M_{s2} I_2(s)$$

Let's consider open-circuit conditions, $I_s(s) = 0$,

$$V_s(s) = j\omega M_{1s} I_1(s)$$

Inserting into expression for $I_s(s)$ we get

$$I_s(s) = \frac{j\omega M_{1s} I_1(s)}{j\omega L_s + R_s}$$

Now we insert this into the expression for $V_s(s)$

$$V_2(s) = M_{12} j\omega I_1(s) - M_{s2} j\omega \frac{j\omega M_{1s} I_1(s)}{j\omega L_s + R_s}$$

Now we are going to make several assumptions. First wee assume that we are operating at high frequency so that $\omega L_s \gg R_s$. That gives us

$$\begin{aligned} V_2(s) \approx & M_{12} j\omega I_1(s) - M_{s2} j\omega \frac{j\omega M_{1s} I_1(s)}{j\omega L_s} \\ = & j\omega I_1(s) \left[M_{12} - M_{s2} \frac{M_{1s}}{L_s} \right] \end{aligned}$$

Next we assume (according to Ott, 1976) that $L_s = M_{s2}$, which gives us

$$V_2(s) \approx j\omega I_1(s) \left[M_{12} - M_{1s} \right]$$

Next, if we assume that conductor 2 and the shield are much closer together than the distance between either and conductor 1, then $M_{12} \approx M_{1s}$, and we get

$$V_2(s) \approx j\omega I_1(s) \left[M_{12} - M_{12} \right] = 0$$

Expand argument in Northrop section 3.10.2.4

In Figure 3.34A the ground-loop voltage produces an additional input voltage. First, we see that $V_- = 0$. Next, let's calculate V_+. First the node equation at the negative terminal of the voltage source V_s, V_A

$$\frac{V_A}{R_{C2}} + \frac{V_A - V_{GL}}{R_G} = 0$$

$$\frac{R_G V_A + R_{C2} V_A - R_{C2} V_{GL}}{R_{C2} R_G} = 0$$

$$R_G V_A + R_{C2} V_A - R_{C2} V_{GL} = 0$$

$$V_A \left(R_G + R_{C2} \right) = R_{C2} V_{GL}$$

$$V_A = V_{GL}\frac{R_{C2}}{R_G + R_{C2}}$$

Next the voltage on the positive input terminal is

$$V_+ = (V_A + V_S)\,\frac{R_{IN}}{R_S + R_{C1} + R_{IN}}$$

Assuming that $R_{IN} \gg R_S + R_{C1}$, we get

$$V_+ = V_A + V_S$$

The common mode input due to the ground loop, V_{CI}, is the difference between V_+ with V_{GL} present and V_+ with $V_{GL} = 0$,

$$V_{CI} = V_+ - V_+(V_{GL} = 0) = V_A = V_{GL}\frac{R_{C2}}{R_G + R_{C2}}$$

If we now introduce a large impedance, Z_{SG} between ground and the negative terminal of the voltage source V_S, we can modify the equation for V_A to be

$$V'_A = V_{GL}\frac{R_{C2}}{R_G + R_{C2} + Z_{SG}}$$

If we assume that $Z_{SG} \gg R_G + R_{C2}$, then we can re-write

$$V'_A \approx V_{GL}\frac{R_{C2}}{Z_{SG}}$$

and the coherent interference input becomes

$$V'_{CI} = V'_A \approx V_{GL}\frac{R_{C2}}{Z_{SG}}$$

Inserting a large impedance between the negative terminal of the voltage source negative terminal and the local ground partially mitigates the effects of a ground-loop current, **if** the differential amplifier has zero CMRR.

Northrop 3.4

(A)

The approximate root power spectrum, e_{na} is

$$\begin{aligned} e_{na} =& \sqrt{\lim_{f \longrightarrow \infty} S_n(f)} \\ =& \sqrt{\frac{4 \times 10^{-6}}{10^{10}}}\,\frac{\mathrm{V}}{\sqrt{\mathrm{Hz}}} \\ =& 2 \times 10^{-8}\,\frac{\mathrm{V}}{\sqrt{\mathrm{Hz}}} \\ =& 20\,\frac{\mathrm{nV}}{\sqrt{\mathrm{Hz}}} \end{aligned}$$

(B)

$$\begin{aligned}\langle v_o^2\rangle =& \int_0^\infty \frac{4\times10^{-6}}{f^2+10^{10}}df \\ =&4\times10^{-11}\int_0^\infty \frac{10^5}{f^2+10^{10}}df \\ =&4\times10^{-11}\frac{\pi}{2}\end{aligned}$$

Thus

$$\mathrm{RMS}\,(v_o) = \sqrt{\langle v_o^2\rangle} = sqrt4\times10^{-11}\frac{\pi}{2} = 7.9\,\mu\mathrm{V}$$

Northrop 3.11
(A)
The differential equation which desribes the filter is

$$\dot{v}_o = av_o + Kx(t)$$

Laplace transform to obtain the transfer function

$$sv_o(s) = av_o(s) + Kx(s)$$

$$v_o(s)\,[s-a] = Kx(s)$$

$$H(s) = \frac{v_o(s)}{x(s)} = \frac{K}{s-a}$$

The input Gaussian white noise power spectrum is $S_i(f) = e_i^2$. The output noise power spectrum, $S_o(f)$ is

$$\begin{aligned}S_o(f) =& S_i(f)\,|H(j2\pi f)|^2 = S_i(f)\left|\frac{K}{j2\pi f-a}\right|^2 \\ =&\frac{e_i^2K^2}{(j2\pi f-a)(-j2\pi f-a)} \\ =&\frac{e_i^2K^2}{(2\pi f)^2+a^2}\end{aligned}$$

The output mean squared noise is

$$\begin{aligned}\langle v_o^2\rangle &= \int_o^\infty S_o(f)df \\ &= \int_0^\infty \frac{e_i^2K^2}{(2\pi f)^2+a^2}df\end{aligned}$$

Make a coordinate transformation: $x = 2\pi f$, $dx = 2\pi df \Longrightarrow df = \frac{dx}{2\pi}$.

$$\begin{aligned}
\langle v_o^2 \rangle &= \int_0^\infty \frac{e_i^2 K^2}{x^2+a^2}\frac{dx}{2\pi} \\
&= \frac{e_i^2 K^2}{2\pi a}\int_0^\infty \frac{a}{x^2+a^2}dx \\
&= \frac{e_i^2 K^2}{2\pi a}\frac{\pi}{2} \\
&= \frac{e_i^2 K^2}{4a}
\end{aligned}$$

The output mean-squared signal of the single-frequency sine-wave is

$$\langle V_o^2 \rangle = \frac{V_s^2}{2}\frac{K^2}{(2\pi f_o)^2 + a^2}$$

The signal-to-noise ratio is then

$$\begin{aligned}
\mathrm{SNR}_o &= \frac{\langle V_o^2 \rangle}{\langle v_o^2 \rangle} = \frac{V_s^2}{2}\frac{K^2}{(2\pi f_0)^2 + a^2}\frac{4a}{e_i^2 K^2} \\
&= \frac{V_s^2}{e_i^2}\frac{2a}{(2\pi f_0)^2 + a^2}
\end{aligned}$$

(B)
The value of a which maximzes SNR_o occurs where

$$\frac{d\mathrm{SNR}_o}{da} = 0$$

First re-write

$$\mathrm{SNR}_o = \frac{V_s^2}{e_i^2} = \frac{2}{\frac{(2\pi f_0)^2}{a} + a}$$

Now we only need to take the derivative of the denominator

$$0 = \frac{d}{da}\text{denominator} = -\frac{(2\pi f_0)^2}{a^2} + 1$$

$$\frac{(2\pi f_0^2)^2}{a^2} = 1$$

$$a = \pm 2\pi f_0$$

The correct solution is

$$a = 2\pi f_0$$

because the other solution produces a negative value for SNR_o.
(C)
The maximized output SNR_o is

$$\begin{aligned}\mathrm{SNR}_{o,\max} =& \frac{V_s^2}{e_i^2}\frac{2\times 2\pi f_0}{(2\pi f_0)^2 + (2\pi f_0)^2} \\ =& \frac{V_s^2}{e_i^2}\frac{1}{2\pi f_0}\end{aligned}$$

Northrop 3.12
(A)
Let's compute the output voltage for some set of input voltages $\{V_i\}$ to the ideal amplifiers. The output voltage from each amplifier is

$$V_i' = KV_i$$

Next, let's write the node equation at the negative input to the output amplifier.

$$\sum_{i=1}^{N}\frac{V_i'}{R} + \frac{V_o}{\frac{R}{N}} = 0$$

Thus

$$\frac{NV_o}{R} = \sum_{i=1}^{N}\frac{V_i'}{R} = \sum_{i=1}^{N}\frac{KV_i}{R}$$

$$V_o = \frac{K}{N}\sum_{i=1}^{N}V_i$$

Now we need to find the mean-square signal and noise using the above formula. For the signal, $V_i = V_s$, and we get

$$\langle V_o^2\rangle = \frac{K^2}{N^2}(NV_s)^2 = K^2V_s^2$$

So for the signal we get the same result as for a single amplifier. For the noise, let's use the noise time-series, e_R, and e_{nai}. So at each amplifier input the noise voltage is

$$e_{ii} = e_{nai} + e_{R_s}$$

After passing through the amplifier the voltage is

$$e_{ai} = Ke_{ii} = Ke_{nai} + Ke_{R_s}$$

Next we use the node equation at the negative input port of the amplifier IOA,

$$\sum_{i=1}^{N}\frac{e_{ai}}{R} + \frac{e_o}{\frac{R}{N}} = 0$$

$$\sum_{i=1}^{N}\left[\frac{Ke_{nai}}{R}+\frac{Ke_{R_s}}{R}\right]+\frac{e_o}{\frac{R}{N}}=0$$

$$\begin{aligned}
e_o =& \frac{R}{N}\frac{K}{R}\left[\sum_{i=1}^{N}e_{nai}+Ne_{R_s}\right]\\
=& \frac{K}{N}\left[\sum_{i=1}^{N}e_{nai}+Ne_{R_s}\right]\\
=& K\left[\frac{1}{N}\sum_{i=1}^{N}e_{nai}+e_{R_s}\right]
\end{aligned}$$

Next we compute the output power spectrum as e_o^2. The only terms that are non-zero are the terms e_{nai}^2 and $e_{R_s}^2$.

$$S_o(f)=e_o^2=K^2\left[\frac{1}{N^2}\sum_{i=1}^{N}e_{nai}^2+e_{R_s}^2\right]$$

Since $e_{nai}=e_{naj}\forall i,j$, we can rewrite

$$\begin{aligned}
S_o(f) =& K^2\left[\frac{1}{N^2}Ne_{na}^2+e_{R_s}^2\right]\\
=& K^2\left[\frac{e_{na}}{N}+e_{R_s}^2\right]\\
=& K^2\left[\frac{e_{na}}{N}+4kTR_s\right]
\end{aligned}$$

Now we can calculate the SNR as

$$\text{SNR} = \frac{V_o^2}{S_oB}=\frac{K^2V_s^2}{K^2\left[\frac{e_{na}}{N}+4kTR_s\right]B}=\frac{V_s^2}{\left[\frac{e_{na}}{N}+4kTR_s\right]B}$$

(B) In the case when the resistors R and R/N also produce noise, e_Ri, and $e_{R/N}$, we can write the node equation as

$$\sum_{i=1}^{N}\left[\frac{e_{ai}}{R}+\frac{e_{Ri}}{R}\right]+\frac{e_o}{\frac{R}{N}}+\frac{e_{R/N}}{R/N}=0$$

$$\sum_{i=1}^{N}\left[\frac{Ke_{nai}}{R}+\frac{Ke_{R_s}}{R}+\frac{e_{R_i}}{R}\right]+\frac{e_o}{\frac{R}{N}}+\frac{e_{R/N}}{\frac{R}{N}}=0$$

$$
\begin{aligned}
e_o =& \frac{R}{N} \sum_{i=1}^{N} \left[\frac{K e_{nai}}{R} + \frac{K e_{R_s}}{R} + \frac{e_{R_i}}{R} \right] - e_{R/N} \\
=& \frac{K}{N} \sum_{i=1}^{N} [e_{nai} + e_{R_s}] + \frac{1}{N} \sum_{i=1}^{N} e_{R_i} - e_{R/N} \\
=& \frac{K}{N} \left[\sum_{i=1}^{N} e_{nai} + N e_{R_s} \right] + \frac{1}{N} \sum_{i=1}^{N} e_{R_i} - e_{R/N}
\end{aligned}
$$

The power spectrum is e_o^2, and when we square, only the terms e_{nai}^2, $e_{R_s}^2$, $e_{R_i}^2$, and $e_{R/N}^2$ are non-zero

$$
S_o(f) = e_o^2 = \frac{K^2}{N^2} \sum_{i=1}^{N} e_{nai}^2 + \frac{K^2}{N^2} N^2 e_{R_s}^2 + \frac{1}{N^2} \sum_{i=1}^{N} e_{Ri}^2 + e_{R/N}^2
$$

Since $e_{nai}^2 = e_{naj}^2 \forall i, j$ and $e_{Ri}^2 = e_{Rj}^2 \forall i, j$, we can rewrite as

$$
\begin{aligned}
S_o(f) =& \frac{K^2}{N^2} N e_{na}^2 + \frac{K^2}{N^2} N^2 e_{R_s}^2 + \frac{1}{N^2} N e_R^2 + e_{R/N}^2 \\
=& K^2 \left[\frac{e_{na}^2}{N} + e_{R_s}^2 \right] + \frac{1}{N} e_R^2 + e_{R/N}^2
\end{aligned}
$$

Now the SNR becomes

$$
\begin{aligned}
\text{SNR} =& \frac{K^2 V_s^2}{S_o(f) B} \\
=& \frac{K^2 V_s^2}{\left(K^2 \left[\frac{e_{na}^2}{N} + e_{R_s}^2 \right] + \frac{1}{N} e_R^2 + e_{R/N}^2 \right) B} \\
=& \frac{V_s^2}{\left(\frac{e_{na}^2}{N} + e_{R_s}^2 + \frac{e_R^2}{N K^2} + \frac{e_{R/N}^2}{K^2} \right) B} \\
=& \frac{V_s^2}{\left(\frac{e_{na}^2}{N} + 4kTR_s + \frac{4kTR}{NK^2} + \frac{4kT\frac{R}{N}}{K^2} \right) B} \\
=& \frac{V_s^2}{\left(\frac{e_{na}^2}{N} + 4kTR_s + \frac{2}{NK^2} 4kTR \right) B}
\end{aligned}
$$