EE 521 Instrumentation and Measurements
Fall 2007
Solutions for homework assignment #2

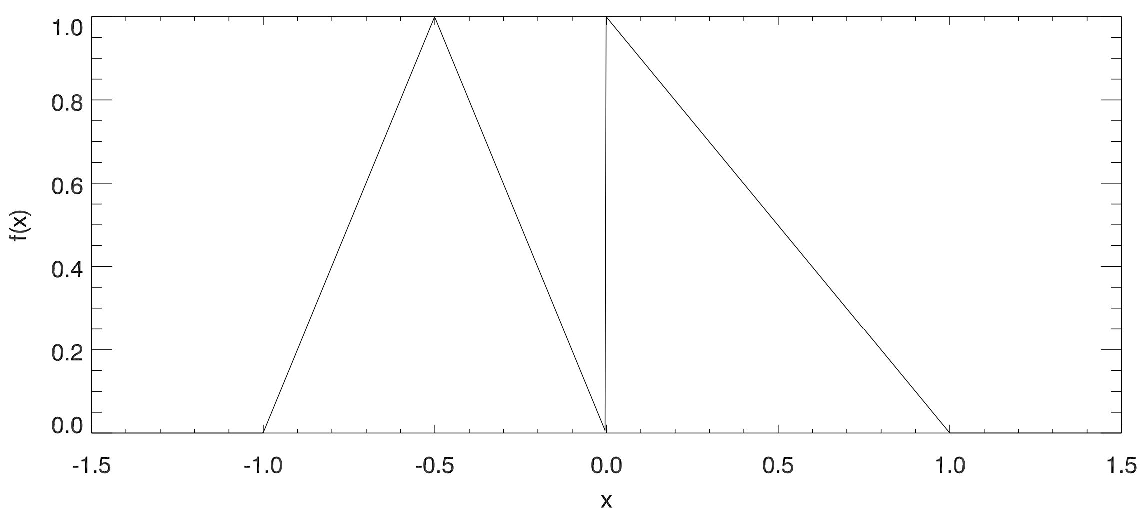
Problem 1
(1)

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

(2)
If the height of the peaks in the distribution as drawn are assumed to be 1, then the area under the curve is $A = 2 \times \frac{0.5\times 1}{2} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$. The function is normalized if the height of the peaks is 1. We can then write

$$f(x) = \begin{cases} 0 & x < -1 \\ 2(x+1) & -1 \leq x < -0.5 \\ -2x & -0.5 \leq x < 0 \\ 1-x & 0 \leq x < 1 \\ 0 & 1 \leq x \end{cases}$$

Here is a plot of $f(x)$,



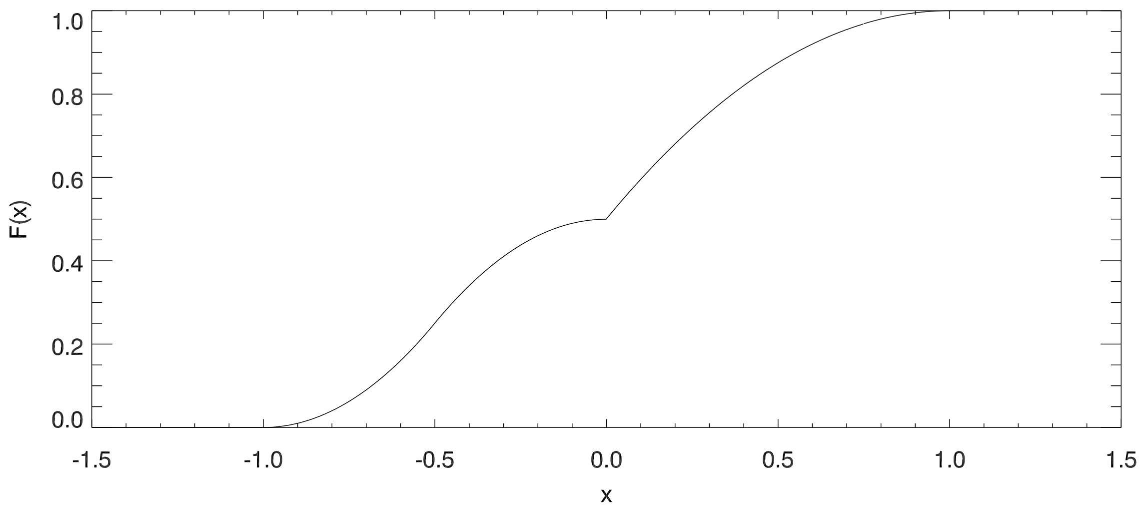
(3)
We first start by writing down the cummulative distribution,

$$F(x) = \int_{-\infty}^{x} f(x)dx$$

It looks like this

$$F(x) = \begin{cases} 0 & x < -1 \\ x^2 + 2x + 1 & -1 \leq x < -0.5 \\ -x^2 + \frac{1}{2} & -0.5 \leq x < 0 \\ -\frac{1}{2}x^2 + x + \frac{1}{2} & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases}$$

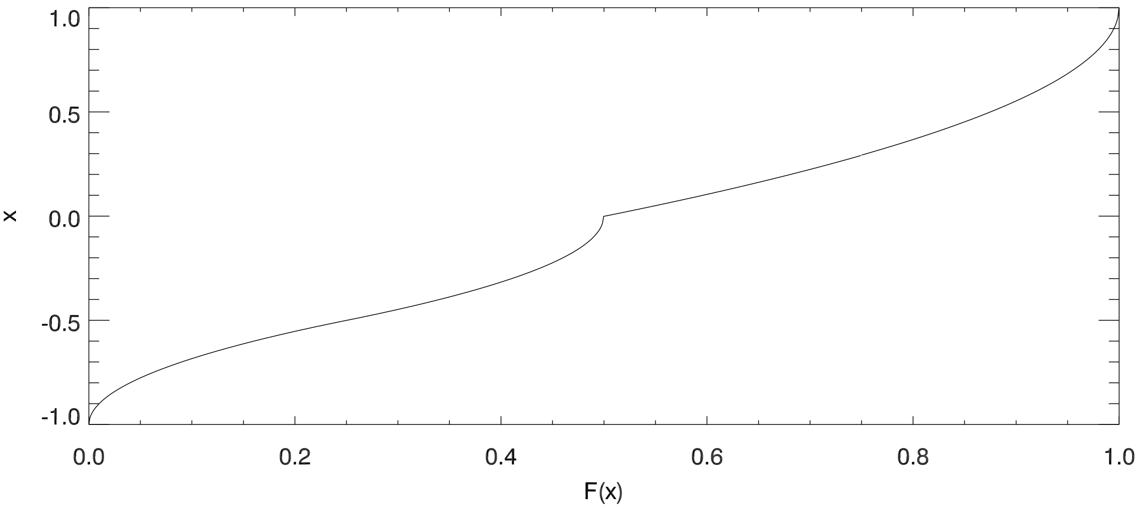
Here is a plot of the cummulative distribution.



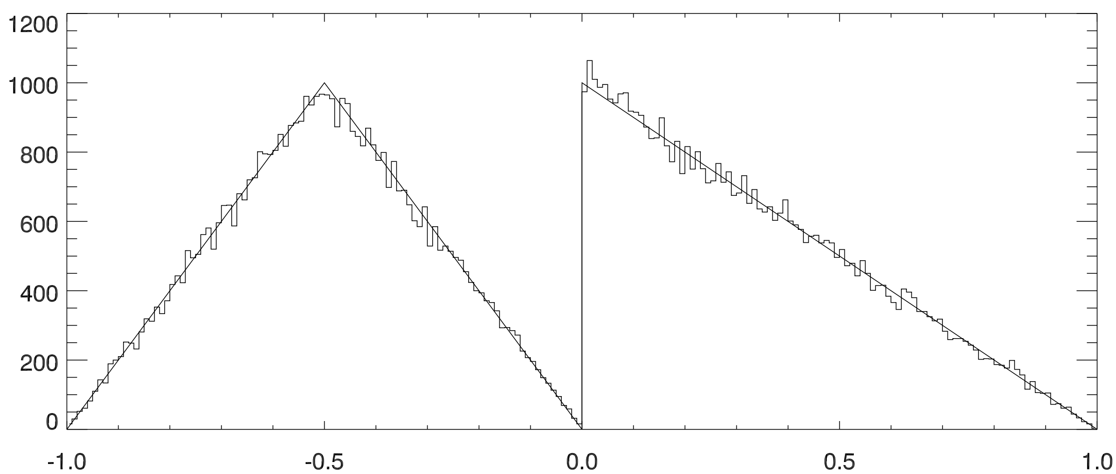
Next we need to solve the above equations for x. So for example the first interval corresponds to $F(x) < \left(-\frac{1}{2}\right)^2 - 1 + 1 = \frac{1}{4}$. In that case, we solve for $x^2 + 2x + 1 = y$ for x as $x = -1 + \sqrt{F(x)}$ (where I selected the sign which would place x between -1 and $-\frac{1}{2}$. We can do the same thing in the second interval and in the third interval and arrive at the piecewise equation

$$x\left(F(x)\right) = \begin{cases} -1 + \sqrt{F(x)} & 0 \leq F(x) < \frac{1}{4} \\ -\sqrt{\frac{1}{2} - F(x)} & \frac{1}{4} \leq F(x) < \frac{1}{2} \\ 1 - \sqrt{2 - 2F(x)} & \frac{1}{2} \leq F(x) \leq 1 \end{cases}$$

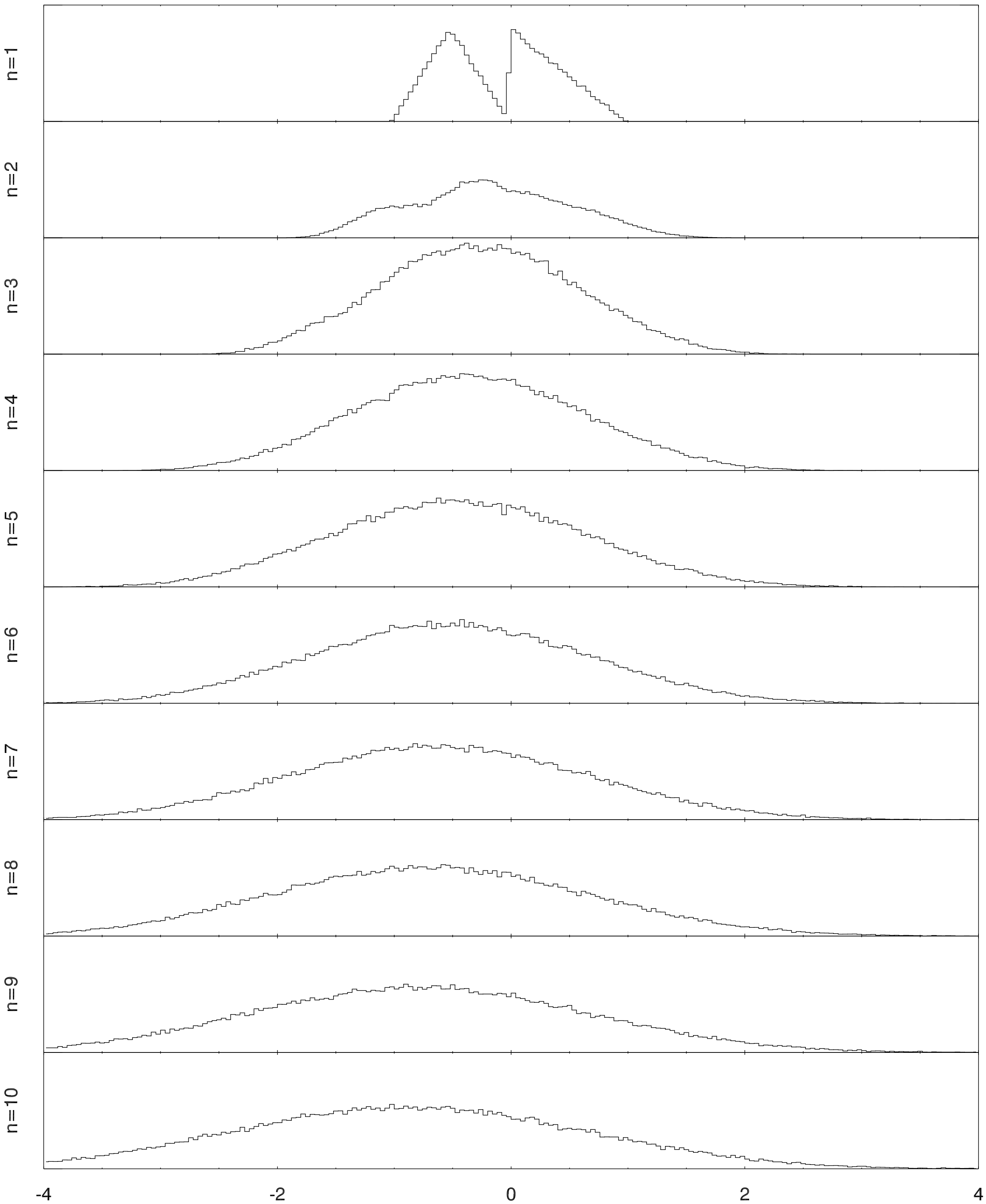
and x undefined for other values of $F(x)$. Here is a plot of the inverse function



(4)
Here is a histogram of 10^5 values using a program which computes the inverse equation.

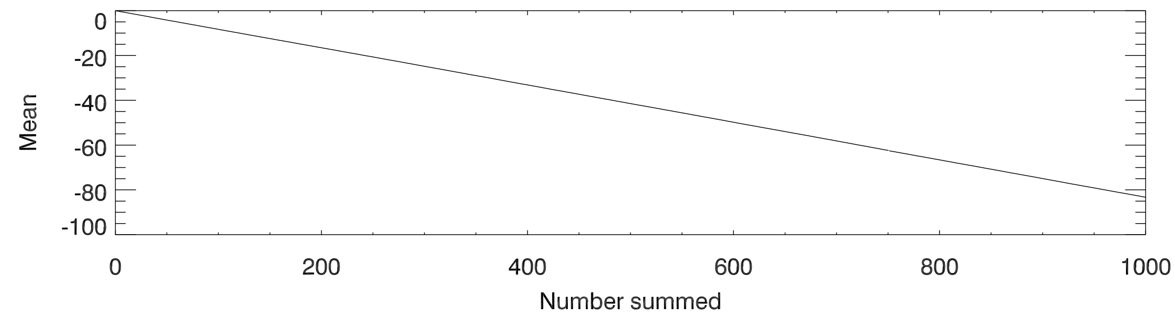


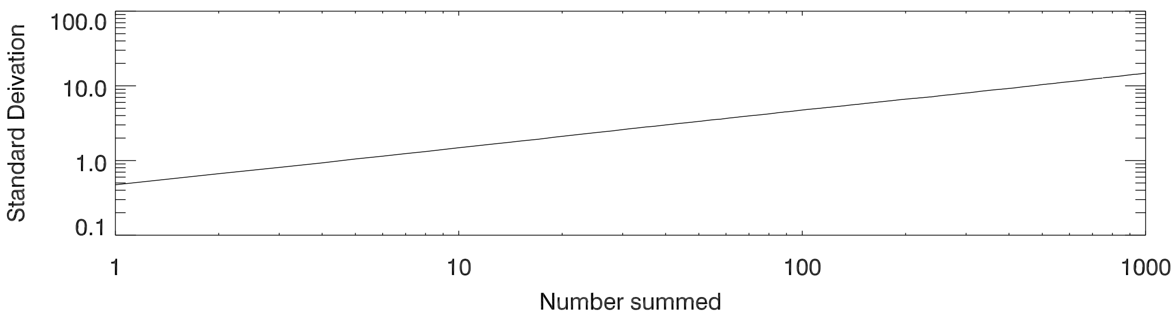
(5)
Here are histograms illustrating what the distribution of sums look like for $n = 1$ to $n = 10$.



(6)

I will compute the mean and standard deviation for the distributions from $n = 1$, to $n = 100$. Those are plotted here





The second plot shows that the standard deviation increases as the square-root of the number of values combined,

$$\sigma_N = \sigma_1 \sqrt{N}$$

Taylor 3.47

We wish to do a error calculation on the expression

$$a = g\frac{M - m}{M + m}$$

using the error propagation formula

$$\delta a^2 = \left(\frac{\partial a}{\partial M}\right)^2 \delta M^2 + \left(\frac{\partial a}{\partial m}\right)^2 \delta m^2$$

Derivative with respect to M,

$$\frac{\partial a}{\partial M} = g\frac{(M + m)1 - (M - m)1}{(M + m)^2} = \frac{2m}{(M + m)^2}$$

And with respect to m,

$$\frac{\partial a}{\partial m} = g\frac{-(M + m) - (M - m)}{(M + m)^2} = \frac{-2M}{(M + m)^2}$$

The expression for δa is then

$$\begin{aligned}\delta a =& g\sqrt{\left(\frac{2m}{(M+m)^2}\right)^2 \delta M^2 + \left(\frac{2M}{(M+m)^2}\right)^2 \delta m^2} \\ =& g\frac{2}{(M+m)^2}\sqrt{m^2\delta M^2 + M^2\delta m^2}\end{aligned}$$

Inserting the values $M = 100 \pm 1$, and $m = 50 \pm 1$, we get

$$\delta a = \frac{2}{(100+50)^2}\sqrt{50^2 + 100^2} = 0.0099g$$

Taylor 3.48

We are to perform error calculations on the function

$$q = \frac{x+y}{x+z}$$

(a)

Assume δy and δz are negligible, then

$$\delta q = \left|\frac{\partial q}{\partial x}\right| \delta x$$

We then get

$$\delta q = \left|\frac{(x+z)-(x+y)}{(x+z)^2}\right| \delta x = \left|\frac{z-y}{(x+z)^2}\right| \delta x$$

Inserting $x \pm \delta x = 20 \pm 1$, $y = 2$, and $z = 0$ we get

$$\delta q = \frac{2}{20^2} = 0.005$$

(b)

Inserting $x \pm \delta x = 20 \pm 1$, $y = -40$, and $z = 0$ we get

$$\delta q = \frac{40}{400} = 0.1$$

The difference:

When $z = 0$, the expression for q becomes

$$q = \frac{x+y}{x} = 1 + \frac{y}{x}$$

Thus the uncertainty in q is proportional to y when only x has uncertainty attached to it.

Taylor 3.50

We will perform an error calculation on the expression

$$q = \frac{x+2}{x+y\cos 4\theta}$$

when x, y, and θ have uncertainty attached to them. We will use the general formula for error propagation,

$$\delta q^2 = \left(\frac{\partial q}{\partial x}\right)^2 \delta x^2 + \left(\frac{\partial q}{\partial y}\right)^2 \delta y^2 + \left(\frac{\partial q}{\partial \theta}\right)^2 \delta \theta^2$$

Note, that in a trigonometric function the unit of the angles must be radians. We can still use degrees, but we will then need to replace the angle θ in the expressin by $\theta\frac{\pi}{180}$, which will then in turn change the terms in the uncertainty equation to balance the units.

First the partial derivative with respect to x,

$$\begin{aligned}\frac{\partial q}{\partial x} =& \frac{(x + y\cos 4\theta) - (x+2)}{(x + y\cos 4\theta)^2} \\ =& \frac{y\cos 4\theta - 2}{(x + y\cos 4\theta)^2}\end{aligned}$$

Next the partial derivative with respect to y

$$\frac{\partial q}{\partial y} = \frac{-(x+2)\cos 4\theta}{(x + y\cos 4\theta)^2}$$

Finally the derivative with respect to θ

$$\frac{\partial q}{\partial \theta} = \frac{(x+2)4y\sin 4\theta}{(x + y\cos 4\theta)^2}$$

Inserting the values $x = 10 \pm 2$, $y = 7 \pm 1$, and $\theta = 40 \pm 3^\circ = 0.6981 \pm 0.0524$ we get

$$q = 3.506$$

$$\frac{\partial q}{\partial x} = -0.7325 \qquad \frac{\partial q}{\partial y} = 0.9627 \qquad \frac{\partial q}{\partial \theta} = 9.8145$$

and

$$\begin{aligned}\delta q =& \sqrt{0.7325^2 \times 2^2 + 0.9627^2 \times 1^2 + 9.8145^2 \times 0.0524^2} \\ =& 1.83\end{aligned}$$

A reasonable answer for q might then be

$$q = 3.5 \pm 1.8$$

Taylor 4.28
(a)

Experiment	1	2	3	4	5
Length, l (cm)	51.2	59.7	68.2	79.7	88.3
Period, T (s)	1.448	1.566	1.669	1.804	1.896
g	964.0	961.1	966.6	966.8	969.7

Next we compute the mean, $\bar{g} = 965.6$ and the standard deviation, $\sigma_g = 3.24$.

The standard deviation of the mean is then

$$\sigma_{\bar{g}} = \frac{\sigma_g}{\sqrt{n}} = \frac{3.24}{\sqrt{5}} = 1.45$$

The measured value for g is then

$$\bar{g} = 965.6 \pm 1.5 \text{ cm/s}^2$$

which is not consistent with the accepted value of 979.6 cm/s^2

(b)

The discrepancy is $979.6 - 965.6 = 14.0$, which is roughly 10 times the expected maximum discrepancy of 1.5.

(c)

If we are certain that the measurement of T has no error in it, then the error must be in l. Since the measured value of g is $14.0/979.6 = 0.014 = 1.4\%$ smaller than the accepted value, it is likely that the measurement of the length of the pendulum is off by 1.5%, which corresponds roughly to $60\,\text{cm} \times 1.5\% = 0.9\,\text{cm}$.

(d)

To correct the data for the systematic effect of the finite radius of the ball, we simply add 1 cm to every length measurement. The data then look like this

Experiment	1	2	3	4	5
Length, l (cm)	52.2	60.7	69.2	80.7	89.3
Period, T (s)	1.448	1.566	1.669	1.804	1.896
g	982.7	977.2	980.7	978.9	980.7

The mean value is $\bar{g} = 980.0$, the standard deviation is $\sigma_g = 2.1$, and the standard deviation of the mean is $\sigma_{\bar{g}} = 0.93$, so we report

$$\bar{g} = 980.0 \pm 0.9 \text{ cm/s}^2$$

The discrepancy from the accepted value is $980.0 - 979.6 = 0.4$, which is within the uncertainty on the measured value.

Taylor 5.21

We have a normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-x_0)^2}{2\sigma^2}}$$

with $x_0 = 5\text{ft}5\frac{1}{2}\text{in} = 65.5\text{in}$, and $\sigma = 2\frac{1}{2}\text{in}$.

(a)

We wish to know which fraction of the distribution lies above $x = 5\text{ft}10\text{in} = 70\text{in}$. That is

$$P(x \geq 5\text{ft}10\text{in}) = \int_{5\text{ft}10\text{in}}^{\infty} f(x)dx = \frac{1}{2} - \int_{x_0}^{5\text{ft}10\text{in}} f(x)dx$$

This integral can be done numerically, and we get

$$P(x \geq 5\text{ft}10\text{in}) = \frac{1}{2} - 0.4641 = 0.0359$$

Since there are 2000 women in the town, approximately $2000 \times 0.0359 = 72$ are elligible to join.

(b)

Change the height requirement such that twice as many women are elligible to join. Now we wish to find x_1, such that

$$P(x \geq x1) = 2 \times P(x \geq 5\text{ft}10\text{in}) = 0.0719$$

This means finding x_1 such that

$$\int_{x_0}^{x_1} f(x)dx = \frac{1}{2} - 0.0719 = 0.4281$$

I can integrate numerically until that value is achieved, and the answer is

$$x_1 = 69.15\text{in} = 5\text{ft}9.15\text{in}$$

To the nearest half inch, this is

$$x_1 = 5\text{ft}9\text{in}$$