EE 521 Instrumentation and Measurements
Fall 2007
Solutions for homework assignment #5

3.11

(A)

The filter is described by the differential equation

$$\dot{v}_o(t) = av_o(t) + Kx(t)$$

Let's Laplace transform it to get

$$sV_o(s) = aV_o(s) + KX(s)$$

Re-write to get the transfer function

$$\frac{V_o(s)}{X(s)} = \frac{K}{s-a}$$

The square modulus of the transfer function is

$$\begin{aligned}\left|\frac{V_o(j\omega)}{X(j\omega)}\right|^2 &= \left|\frac{K}{j\omega - a}\right|^2 \\ &= \frac{K^2}{(j\omega - a)(-j\omega - a)} \\ &= \frac{K^2}{\omega^2 + a^2}\end{aligned}$$

We are also given that

$$x(t) = n(t) + V_s \sin(\omega_o t)$$

where $n(t)$ is the white noise with power spectrum e_n^2. First we pass the signal through this filter to get the MS output signal

$$\langle V_o^2 \rangle = \frac{V_s}{2} \frac{K^2}{\omega_0^2 + a^2}$$

Next we pass the noise through the filter to get the MS output noise

$$
\begin{aligned}
\langle v_o^2 \rangle = &\int_0^\infty e_n^2 \frac{K^2}{\omega^2 + a^2} d\omega \\
=& e_n^2 K^2 \int_0^\infty \frac{1}{\omega^2 + a^2} d\omega \\
=& e_n^2 K^2 \left[\frac{1}{a} \tan^{-1} \frac{x}{a} \right]_0^\infty \\
=& \frac{e_n^2 K^2}{a} \left[\tan^{-1} \infty - \tan^{-1} 0 \right] \\
=& \frac{e_n^2 K^2}{a} \frac{\pi}{2}
\end{aligned}
$$

The SNR is then

$$
\text{SNR} = \frac{\langle V_o^2 \rangle}{\langle v_o^2 \rangle} = \frac{a V_s^2}{\pi e_n^2} \frac{1}{\omega_0^2 + a^2}
$$

(B)
First re-write the expression for SNR as

$$
\text{SNR} = \frac{V_s^2}{\pi e_n^2} \frac{1}{\frac{\omega_0^2}{a} + a}
$$

Next find the value of a which minimizes the denominator

$$
\frac{d}{da} \left(\frac{\omega_0^2}{a} + a \right) = 0 = -\frac{\omega_0^2}{a^2} + 1
$$

or

$$
a = \omega_0
$$

(C)
The maximum output SNR is when $a = \omega_0$,

$$
\text{SNR} = \frac{\omega_0 V_s^2}{2\pi e_n^2 \omega_0^2} = \frac{V_s^2}{2\pi e_n^2 \omega_0}
$$

3.12
Immediately after the resistor R_s, the signal is

$$
V = V_s
$$

and the noise is

$$
v = e_R
$$

Immediately before each amplifier the signal is the same, and the noise is

$$v_i = e_R + e_i$$

Immediately after each amplifier the signal is

$$V_i = KV_s$$

and the noise is

$$v_i = K\left(e_R + e_i\right)$$

The signal current flowing to the input of the IOA is

$$I_i = \frac{KV_s}{R}$$

and the noise current is

$$i_i = fracK\left(e_R + e_i\right)R$$

The signal current flowing through the resistor R/N is

$$I_o = \sum_{i=1}^{N} \frac{KV_s}{R} = \frac{KV_s N}{R}$$

and the noise curent flowing through the resistor R/N is

$$i_o = \sum_{i=1}^{N} \frac{K\left(e_R + e_i\right)}{R}$$

now think carefully about how this sum plays out:

$$i_o = \frac{NKe_R}{R} + \frac{K}{R}\sum_{i=1}^{N} e_i$$

The signal voltage at the output is

$$V_o = I_o \frac{R}{N} = KV_s$$

and its mean squared value is

$$\langle V_o^2 \rangle = K^2 \langle V_s^2 \rangle$$

The noise voltage at the output is

$$v_o = i_o \frac{R}{N} = Ke_R + \frac{K}{N}\sum_{i=1}^{N} e_i$$

and its mean squared value is

$$\langle v_o \rangle = K^2 \left(e_R^2 + \frac{1}{N^2} \sum_{i=1}^{N} e_i^2 \right)$$

The SNR at the output is then

$$\frac{\langle V_o^2 \rangle}{\langle v_o^2 \rangle} = \frac{\langle V_s^2 \rangle}{e_R^2 + \frac{1}{N^2} \sum_{i=1}^{N} e_i^2} = \frac{\langle V_s^2 \rangle}{e_R^2 + \frac{e_{na}^2}{N}} = \frac{\langle V_s^2 \rangle}{4kTR + \frac{e_{na}^2}{N}}$$

(B)

If we include noise in the resistors R, and R/N, there is not change in $\langle V_o^2 \rangle$. The noise voltage after the resistors becomes

$$v_i = K(e_R + e_i) + e_{Ri}$$

with the total noise current at the IOA input

$$i = \frac{1}{R} \left[K(N e_R + \sum_{i=1}^{N} e_i) + e_{Ri} \right]$$

At the output the noise voltage becomes

$$v_o = i \frac{R}{N} + e_{R/N} = K e_R + \frac{1}{N} \sum_{i=1}^{N} (K e_i + e_{Ri}) + e_{R/N}$$

The MS noise output is then

$$\begin{aligned}
\langle v_o^2 \rangle =& K^2 e_R^2 + \frac{1}{N^2} \sum_{i=1}^{N} \left(K^2 e_i^2 + e_{Ri}^2 \right) + e_{R/N}^2 \\
=& K^2 e_R^2 + \frac{1}{N} \left(K^2 e_{na}^2 + e_R^2 \right) + e_{R/N}^2 \\
=& K^2 \left[\left(1 + \frac{1}{NK^2} \right) e_R^2 + \frac{e_{na}^2}{N} + \frac{e_{R/N}}{K^2} \right] \\
=& K^2 \left[4kTR \left(1 + \frac{2}{NK^2} \right) + \frac{e_{na}^2}{N} \right]
\end{aligned}$$

The MS SNR is then

$$\text{SNR} = \frac{\langle V_o^2 \rangle}{\langle v_o^2 \rangle} = \frac{\langle V_s^2 \rangle}{4kTR \left(1 + \frac{2}{NK^2} \right) + \frac{e_{na}^2}{N}}$$

So we reduce the amplifier noise by N, and only add a little $(\frac{2}{NK^2})$ resistor noise.