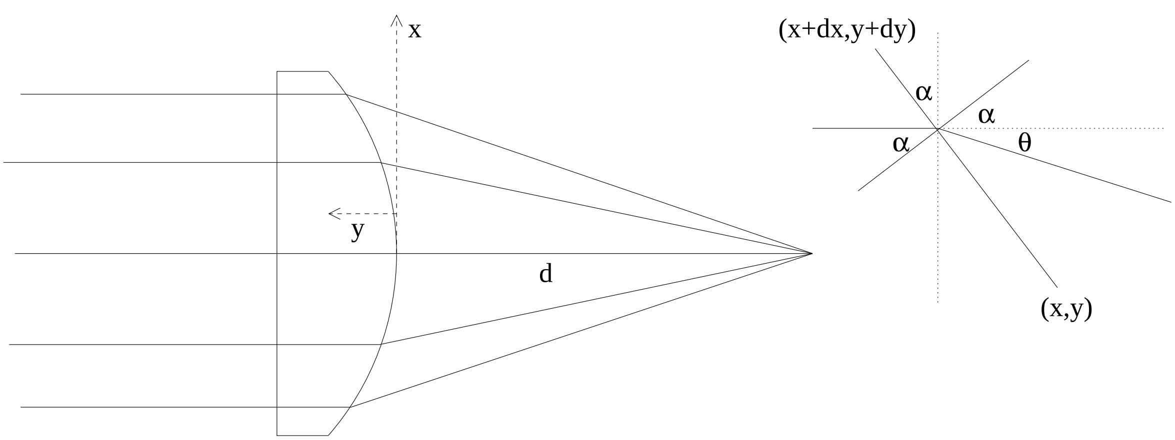
Solutions to homework #2 due 2007/2/9

Questions:

1. Calculate the differential equation dx/dy for the back surface which focuses light rays that enter the lens parallel to the optical axis perfectly at some distance d from the back end of the lens. y is the distance from the optical axis, and x is the distance from the back of the lens.
2. Integrate the expression to compute the shape of the lens. You are permitted to make the approximation that d is very large.
3. Are there other lens shapes that could focus the same set of rays perfectly? You do not need to derive the equation, just draw an example illustrating one such lens.
4. Show that the lens you derived the shape of in 2) does not focus rays that enter at an angle relative to the optical axis. You can do this by sending three rays through the lens and show that they do not meet in a point.

Answers:

1. This figure shows the lens and the quantities which we will use to evaluate its shape. The left portion of the figure shows the coordinate systems, whereas the right side of the figure shows a small portion of the lens around where a ray passes through it.



We can then write

$$\tan\theta = \frac{x}{y+d} = \frac{\sin\theta}{\cos\theta}$$

$$\frac{dy}{dx} = \tan\alpha$$

The write Snell's law

$$n\sin\alpha = \sin(\alpha+\theta) = \sin\alpha\cos\theta + \cos\alpha\sin\theta$$

Divide by $\cos\alpha$ on both sides

$$n\tan\alpha = \tan\alpha\cos\theta + \sin\theta$$

But $\tan\alpha = \frac{dy}{dx}$. Insert it

$$n\frac{dy}{dx} = \frac{dy}{dx}\cos\theta + \sin\theta$$

$$\frac{dy}{dx}\left(n - \cos\theta\right) = \sin\theta$$

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$$\frac{dy}{dx} = \frac{\sin\theta}{n - \cos\theta}$$

We should insert the expressions for $\sin\theta$ and $\cos\theta$, which are

$$\cos\theta = \frac{d+y}{\sqrt{\left(d+y\right)^2 + x^2}} \qquad \sin\theta = \frac{x}{\sqrt{\left(d+y\right)^2 + x^2}}$$

So that

$$\frac{dy}{dx} = \frac{\frac{x}{\sqrt{\left(d+y\right)^2+x^2}}}{n - \frac{d+y}{\sqrt{\left(d+y\right)^2+x^2}}} = \frac{x}{n\sqrt{\left(d+y\right)^2 + x^2} - d + y}$$

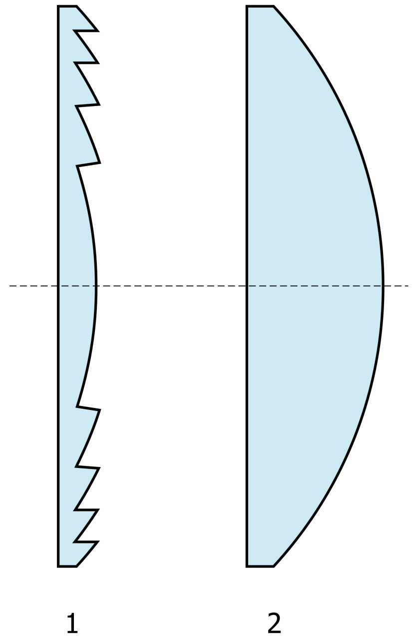
2. The expression is difficult to integrate, so let us assume that $d \gg$ and $d \gg x$. In that case we can write

$$\frac{dy}{dx} \approx \frac{x}{n\sqrt{d^2} - d} = \frac{x}{d}\frac{1}{n-1}$$

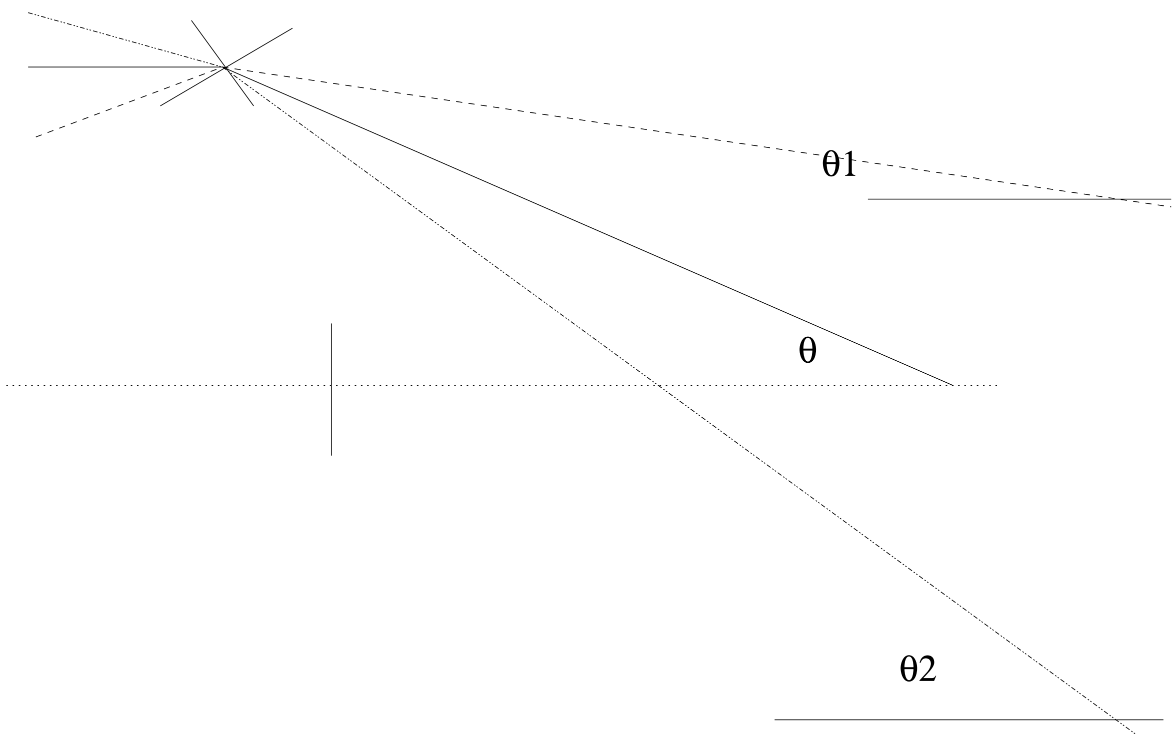
Integrating it we get

$$y(x) = \int_0^{y(x)} dy = \frac{1}{d}\frac{1}{n-1}\int_0^x xdx = \frac{x^2}{2d\left(n-1\right)}$$

3. Yes, there are many. This was a plano-convex lens. There are also bi-convex lenses which can focus light. An interesting alternative is the fresnel lens



4. Consider the following figure. We have drawn 3 rays, the solid entering parallel to the optical axis, the dashed making an angle $-\beta$ with the optical axis and the dot-dashed making an angle β with the optical axis.



The ray which makes the angle β corresponds to a parallel ray entering at the symmetrical point around the optical axis at an angle $-\beta$. So in order for the lens to still focus light properly, it must be true that

$$\theta_1 - \theta = \theta - \theta_2$$

If we write the expressions

$$n \sin(\alpha + \beta) = \sin(\alpha + \theta_1) \qquad n \sin(\alpha - \beta) = \sin(\alpha + \theta_2)$$

It is clear that this is only satisfied for $\beta = 0$.