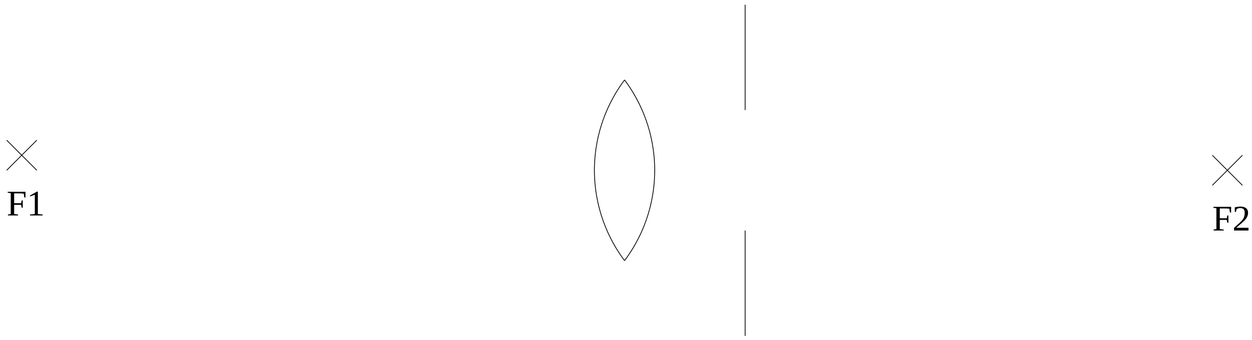
Solutions to homework #7 due 2007/4/3

Problem 1



The diaphragm is the limiting aperture (rays from an axial object to the left of F_1 will hit the edge of the diaphragm before they hit the edge of the lens). The entrance pupil is the image of the aperture in object space. To find it, write

$$\frac{1}{s'} = \frac{1}{s} + \frac{1}{f}$$

Since we are propagating rays from right to left we need to change the signs of s' and s so we get

$$-\frac{1}{s'} = -\frac{1}{s} + \frac{1}{f}$$

with $s = 20\,\text{mm}$, and $f = 100\,\text{mm}$, so

$$s' = -\frac{1}{-\frac{1}{20} + \frac{1}{100}} = 25\,\text{mm}$$

So the entrance pupil is located 25 mm to the right of the lens. The size of the aperture can be derived from the formula

$$m = \frac{h'}{h} = \frac{s'}{s}$$

$$h' = h\frac{s'}{s} = 10\,\text{mm}\frac{25}{20} = 12.5\,\text{mm}$$

The exit pupil is the image of the aperture in image space. It is the same as the aperture itself, so it is 20 mm to the right of the lens and has a diameter of 10 mm.

Problem 2

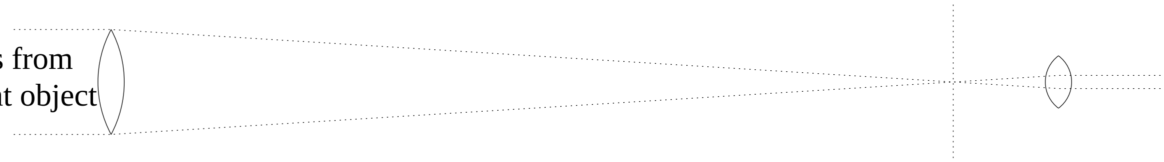
The f-ratio of an optical system is the focal length divided by the diameter of the entrance pupil. When light comes from the left the entrance pupil is 12.5 mm (it is the entrance pupil that we computed in problem 1), and we get

$$\text{f-ratio} = \frac{100}{12.5} = 8$$

If light enters from the right, the entrance pupil is equal to the exit pupil computed in problem 1, so we get

$$\text{f-ratio} = \frac{100}{10} = 10$$

Problem 3



(a) The figure is drawn to scale. It can be seen that the limiting aperture is the front lens, which is also called the objective lens. Thus, the entrance pupil is at the objective lens and has the same diameter as the objective lens, 1". The exit pupil is the image of the aperture. The aperture is the objective lens.

$$\frac{1}{s'} = \frac{1}{s} + \frac{1}{f}$$

$$s' = \frac{1}{\frac{1}{s} + \frac{1}{f}} = \frac{1}{\frac{1}{-11} + \frac{1}{1}} = 1.1$$

The diameter of the exit pupil, h', is then

$$h' = h\frac{s'}{s} = 1\text{"} \times \frac{1.1}{11} = 0.1\text{"}$$

(b) The un-vignetted entrance field of view is the entrance angle of a ray at the lower edge of the objective which reaches the upper edge of the eye lens. The angle of such a ray between the two lenses is

$$u_1' = u_2 = \frac{0.75\text{"}}{11\text{"}} = 0.068\bar{1}\bar{8}$$

We then use the formula from Chapter 2,

$$u_1' = u_1 - \frac{y}{f}$$

$$u_1 = u_1' + \frac{y}{f} = 0.068\bar{1}\bar{8} + \frac{-0.5}{10} = 0.\bar{1}\bar{8}$$

The un-vignetted FOV is thus $\pm 0.1818\,\text{radians} = \pm 1.04°$
For complete vignetting we are looking at a ray which comes from the top of the objective lens and hits the top of the eye lens. It has angle

$$u_1' = u_2 = \frac{-0.25\text{"}}{11\text{"}} = -0.022\bar{7}\bar{2}$$

The entrance angle is

$$u_1 = u_1' + \frac{y}{f} = -0.022\bar{7}\bar{2} + \frac{0.5\text{"}}{10\text{"}} = 0.02727\,\text{radians} = 1.56°$$

The completely vignetted FOV is thus $\pm 0.02727\,\text{radians} = \pm 1.56°$.

We can compute the exit FOV by the same formula, or we can simply look at the magnification. The magnification of this telescope is 10. We compress the beam by a factor of 10, which means that we increase the FOV by a factor of 10.

Problem 4

We know that

$$m = \frac{h'}{h} = \frac{s'}{s} = -4$$

$$\frac{f}{D} = 4$$

$$f = 4\,\mathrm{in} \Rightarrow D = 1\,\mathrm{in}$$

We want to find

$$NA = u = \left|\frac{D}{2s}\right| \qquad \text{and} \qquad NA' = u' = \left|\frac{D}{2s'}\right|$$

We can combine

$$\frac{1}{s'} = \frac{1}{s} + \frac{1}{f} \qquad \text{and} \qquad m = \frac{s'}{s}$$

to produce

$$s = \left(\frac{1}{m} - 1\right) f \qquad \text{and} \qquad s' = (1 - m)\, f$$

Inserting those into the equations for NA and NA' above we get

$$NA = \left|\frac{D}{2\left(\frac{1}{m} - 1\right) f}\right| = \left|\frac{1}{2\left(\frac{1}{-4} - 1\right) 4}\right| = 0.1$$

and

$$NA' = \left|\frac{D}{2\,(1 - m)\, f}\right| = \left|\frac{1}{2\,(1 + 4)\, 4}\right| = 0.025$$

Problem 5

This is an application of the $\cos^4$ rule. The illumination of an image point is related as $\cos^4$ of the angle that the ray makes with the optical axis in image space. We first need to find the location of the image plane. Since we are imaging an object at infinity the image plane is located distance $s' = B$ from the system, where B is the back focal distance of the combined system. We first computed the combined focal length, f_{ab}, and then the back focal distance, B.

$$f_{ab} = \frac{f_a f_b}{f_a + f_b - d} = \frac{16 \times 8}{16 + 8 - 8} = 8"$$

$$s' = B = \frac{f_{ab}\,(f_a - d)}{f_a} = \frac{16 \times (16 - 8)}{16} = 4\text{”}$$

The illumination intensity is related to the distance from the axis as

$$\frac{I}{I_0} = \cos^4\theta$$

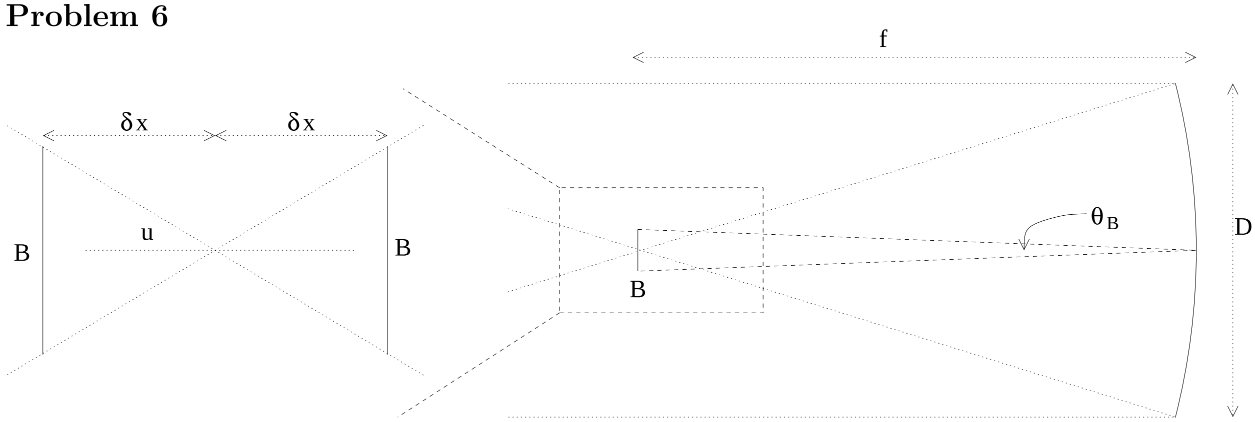
where θ is the angle of the point as seen from the exit pupil. The exit pupil is at the second lens, and the image is 4” from the second lens, and the image point is 3” off the axis, so

$$\theta = \tan^{-1}\left(\frac{y}{B}\right)$$

and the relative illumination is

$$\frac{I}{I_0} = \cos^4\left(\tan^{-1}\left(\frac{y}{B}\right)\right) = \cos^4\left(\tan^{-1}\left(\frac{3}{4}\right)\right) = 0.410$$

Problem 6



We place the reticle at the focus of the mirror. We wish to determine the distance that we can move the reticle such that the blur is acceptable. The information we have is $N = 5$ and $D = 6$”, where

$$N = \frac{f}{D}$$

The acceptable blur is θ_B as seen from the mirror. The size of the blur at the focus is then

$$B = \theta_B f = \theta_B N D$$

The relationship between the depth of focus, δx, and the size of the blur, B, is

$$\frac{B/2}{\delta x} = u = \frac{D/2}{f} = \frac{1}{2N}$$

$$\delta x = BN = \theta_B N^2 D = 0.1 \times 10^{-3} \times 5^2 \times 6\text{”} = 0.015\text{”}$$

If the mirror is $f/2$, then we get

$$\delta x = \theta_B N^2 D = 0.1 \times 10^{-3} \times 2^2 \times 6\text{”} = 0.0024\text{”}$$

Problem 7

For this problem we need to look at depth-of-field. In class we discussed the two formulas

$$s_N = \frac{sf^2}{f^2 - NB\left(s+f\right)} \qquad \text{and} \qquad s_F = \frac{sf^2}{f^2 + NB\left(s+f\right)}$$

If we place the image plane such that an object at distance s is in focus, and the acceptable blue is B in the image plane, and the f-ratio is N, then objects as close as s_N and as far as s_F are also acceptably in focus.

The hyperfocal distance is the distance s, for which $s_F = \infty$. In other words, the denominator of the second formula must be zero, so we have

$$0 = f^2 + NB\left(s_H + f\right)$$

(a) If $s_H = -100$", $f = 10$", and $N = 10$, we wish to determine B

$$B = -\frac{f^2}{N\left(s_H + f\right)} = -\frac{10^2}{10 \times (-100 + 10)} = 0.11\text{"}$$

(b) We insert the number we know into the expression for s_N

$$s_N = \frac{s_H f^2}{f^2 - NB\left(s_H + f\right)} = \frac{100 \times 10^2}{10^2 - 10 \times 0.111\left(-100 + 10\right)} = 50\text{"}$$

(c) Show that if $s = s_H$, then $s_N = \frac{s_H}{2}$. First calculate the value of s_H. From before we have

$$0 = f^2 + NB\left(s_H + f\right)$$

from which we get

$$s_H = -\frac{f^2}{NB} - f = -f\left(\frac{f}{NB} + 1\right)$$

Insert $s = s_H$ into expression for s_N,

$$\begin{aligned}
s_N &= \frac{s_H f^2}{f^2 - NB\left(s_H + f\right)} = \frac{-\left(\frac{f}{NB} + 1\right) f^3}{f^2 - NB\left(-f\left(\frac{f}{NB} + 1\right) + f\right)} \\
&= \frac{-\left(\frac{f}{NB} + 1\right) f^3}{f^2 - NB\left(\frac{-f^2}{NB}\right)} = \frac{-\left(\frac{f}{NB} + 1\right) f^3}{f^2 + f^2} = -\left(\frac{f}{NB} + 1\right) f \times \frac{1}{2} \\
&= \frac{s_H}{2}
\end{aligned}$$

Problem 8

For this problem we must combine the effects of the illumination due to f-ratio and the illumination due to being off-axis in the image plane. Illumination is proportional to the inverse of the square of the f-ratio,

$$I \sim \frac{1}{N^2}$$

The illumination is also proportional to the 4th power of the cosine of the image angle, so

$$I \sim \cos^4 \theta$$

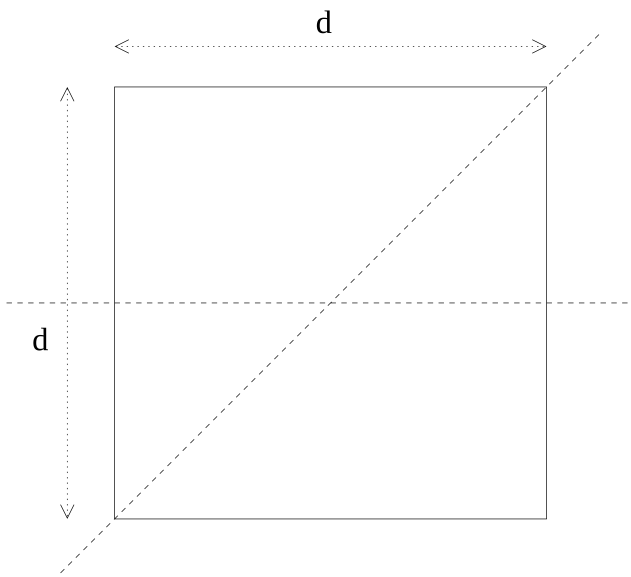
intensity thus scales as

$$I \sim \frac{\cos^4 \theta}{N^2}$$

If we assume the same constant of proportionality for the two systems, we can write

$$\frac{I_{16}}{I_8} = \frac{\cos^4 \theta_{16}}{\cos^4 \theta_8} \frac{N_8^2}{N_{16}^2} = \frac{\cos^4 30°}{\cos^4 45°} \frac{8^2}{16^2} = 0.56$$

Problem 9



The fringe pattern intensity is, from equation 6.16,

$$I = I_0 \frac{\sin^2 m_1}{m_1^2} \cdot \frac{\sin^2 m_2}{m_2^2}$$

with

$$m_i = \frac{\pi d_i \sin \alpha_i}{\lambda}$$

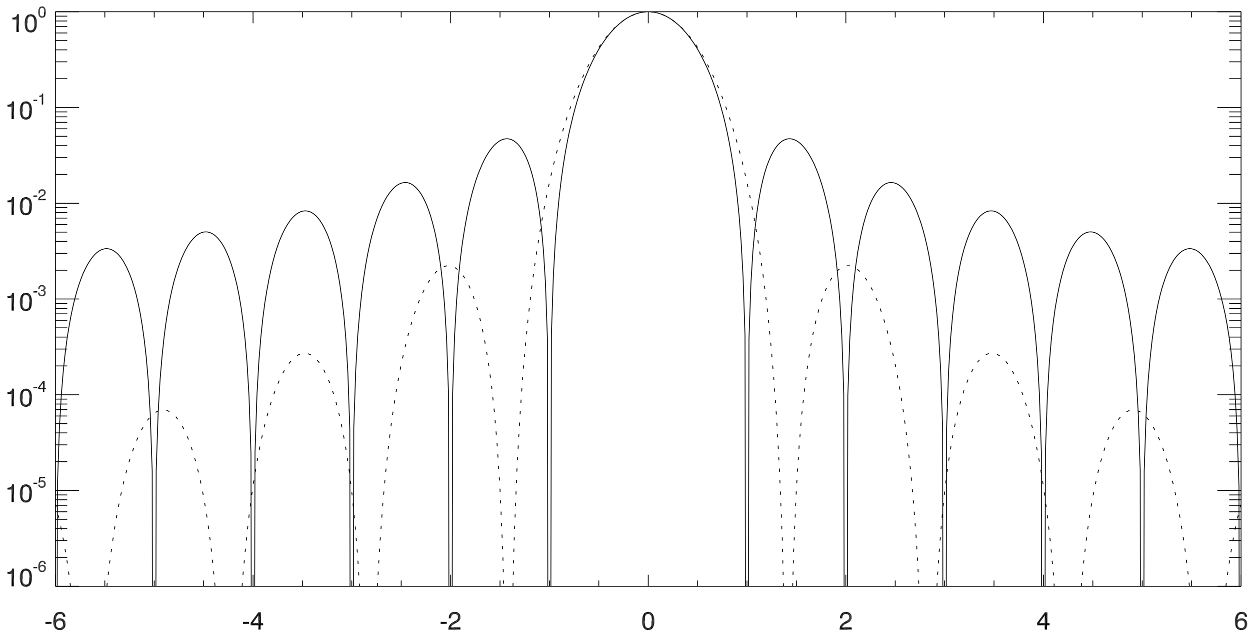
and

$$\tan \alpha_i = \sin \alpha_i = \frac{x_i}{l'} \qquad \text{for} \qquad x_i \ll l'$$

so

$$m_i = \frac{\pi d_i x_i}{l' \lambda} = \frac{\pi x_i}{NA\lambda}$$

we can make the x-axis in terms in units of $NA\lambda$.



Problem 10

We are imaging a distant point, which means that the image plane is located f behind the image system. Let B be the spot size which defines the resolution of the optical system at distance f. The angular resolution is then

$$\theta = 1.22\frac{\lambda}{D} = \frac{B/2}{f}$$

the f-ratio is

$$N = \frac{f}{D}$$

and with the previous expression we can re-write as

$$N = \frac{f}{D} = \frac{B/2}{1.22\lambda} = \frac{0.01\,\mathrm{mm}/2}{1.22 \times 0.00055\,\mathrm{mm}} = 7.45$$

Problem 11

The blur from the hole size, D, is equal to the hole size. The angular blur from diffraction is

$$\theta = 2 \times 1.22\frac{\lambda}{D}$$

(the 1.22 factor is only radius). The spatial blue from diffraction is $d\theta$, so we equate the hole size blur and the spatial diffraction blur

$$D = 2.44\frac{\lambda}{D}d$$

and get

$$D^2 = 2.44\lambda d$$

$$D = \sqrt{2.44\lambda d} = \sqrt{2.44 \times 0.55 \times 10^{-6} \times 0.1} = 0.37\,\text{mm}$$

Problem 12
If they are microscope objectives, the magnification is very large, so the objects are essentially at the first focal point of the objective. Also assume that $\lambda = 0.55\,\mu$m. If the angular resolution is θ, then the spatial resolution at the object is

$$B = \theta f$$

We also have

$$\theta = 1.22\frac{\lambda}{D} \qquad \text{and} \qquad NA = \frac{D/2}{f}$$

so we get

$$D = 2NAf, \qquad \theta = 1.22\frac{\lambda}{2NAf}, B = 1.22\frac{\lambda f}{2NAf} = 1.22\frac{\lambda}{2NA}$$

(a) $NA = 0.25 \Rightarrow B = 0.0013\,\text{mm}$
(b) $NA = 0.8 \Rightarrow B = 0.00042\,\text{mm}$
(c) $NA = 1.2 \Rightarrow B = 0.00028\,\text{mm}$
Problem 13
The resolution of the telescope is

$$\theta = 1.22\frac{\lambda}{D}$$

$$D = 1.22\frac{\lambda}{\theta} = 1.22\frac{0.55\,\mu\text{m}}{11''} = 1.22\frac{0.55 \times 10^{-6}}{5.33 \times 10^{-5}} = 1.26\,\text{cm} = 0.50\,\text{in}$$

The eye can resolve $1'$, so the telescope must magnify by a factor of

$$\frac{1'}{11''} = \frac{60}{11} = 5.45$$

Problem 14
The resolution of a prism is

$$R = \frac{\lambda}{\delta\lambda} = B\frac{dn}{d\lambda}$$

We are given $\frac{dn}{d\lambda} = 0.1\,\mu\text{m}^{-1}$, and $B = 1\,\text{in}$, so we get

$$R = B\frac{dn}{d\lambda} = 0.0254 \times 0.1 \times 10^{6} = 2540$$

The resolution of a grating is

$$R = mN = mnd$$

Where $m = 1$ is the order, $n = 15000\,\mathrm{in}^{-1}$ is the density of lines, and $d = 1\,\mathrm{in}$ is the width of the grating. Inserting we get

$$R = mnd = 1 \times 15000 \times 1 = 15000$$