Lecture 3

Probability and Stochastic Processes

EE 521: Instrumentation and Measurements

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Aly El-Osery, Electrical Engineering Dept., New Mexico Tech

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1 Review Material

1.1 Signal Classification

Assume the voltage across a resistor R is $e(t)$ and is producing a current $i(t)$. The instantaneous power per ohm is $p(t) = e(t)i(t)/R = i^2(t)$.

Total Energy

$$E = \lim_{T\to\infty} \int_{-T}^{T} i^2(t)dt \tag{1}$$

Average Power

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} i^2(t)dt \tag{2}$$

3.3

Arbitrary signal $x(t)$

Total Normalized Energy

$$E \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt \tag{3}$$

Normalized Power

$$P \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \tag{4}$$

- $x(t)$ is an *energy signal* iff $0 < E < \infty$, so that $P = 0$.
- $x(t)$ is a *power signal* iff $0 < P < \infty$, so that $E = \infty$.

3.4

1.2 Time Averages

For Energy Signals

$$\phi(\tau) = \int_{-\infty}^{\infty} x(\lambda)x(\lambda+\tau)df \tag{5}$$

Provides a measure of similarity or coherence between a signal and a delayed version of itself. Note that $\phi(0) = E$

For Power Signals

$$R(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t+\tau)dt \tag{6}$$

For Periodic Signals

$$R(\tau) = \frac{1}{T_0} \int_{T_0} x(t)x(t+\tau)dt \tag{7}$$

3.5

1.3 Frequency Domain

Fourier Transform Equations

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \tag{8}$$

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df \tag{9}$$

3.6

Energy Spectral Density

Rayleigh's Energy Theorem or Parseval's theorem

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df \tag{10}$$

Energy Spectral Density

$$G(f) \triangleq |X(f)|^2 \tag{11}$$

with units of *volts*^{2}-*sec*^{2} or, if considered on a per-ohm basis, *watts-sec/Hz=joules/Hz*

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Power Spectral Density

$$P = \int_{-\infty}^{\infty} S(f)df = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \tag{12}$$

where we define $S(f)$ as the power spectral density with units of watts/Hz. Note that $R(0) = \int_{-\infty}^{\infty} S(f)df$. 3.8

Proof

In an analogy to the energy signals, let us define a function that would give us some indication of the relative power contributions at various frequencies, as $S_x(\omega)$. This function has units of power per Hz and its integral yields the power in $x(t)$ and is known as *power spectral density* function. Mathematically,

$$P = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega)d\omega. \tag{13}$$

Assume that we are given a signal $x(t)$ and we truncate it over the interval $(-T/2, T/2)$. This truncated version is $x(t)\Pi(t/T)$. If $x(t)$ is finite over the interval $(-T/2, T/2)$, then the truncated function $x(t)\Pi(t/T)$ has finite energy and its Fourier transform $X_T(\omega)$ is

$$X_T(\omega) = \mathscr{F}\{x(t)\Pi(t/T)\}. \tag{14}$$

Parseval's theorem of the truncated version is

$$\int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_T(\omega)|^2 d\omega. \tag{15}$$

Therefore, the average power P across a one-ohm resistor is given by

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T} \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_T(\omega)|^2 d\omega. \tag{16}$$

Combining Equations (1) and (4), we get

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega)d\omega = \lim_{T\to\infty} \frac{1}{T} \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_T(\omega)|^2 d\omega. \tag{17}$$

In addition if we insist that this relation should hold over each frequency increment, then

$$M_x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\omega} S_x(u)du = \lim_{T\to\infty} \frac{1}{T} \frac{1}{2\pi} \int_{-\infty}^{\omega} |X_T(u)|^2 du. \tag{18}$$

$M_x(\omega)$ is known as the *cumulative power spectrum*. Now, interchange the order of the limiting operator and the integration (assuming it is valid)

$$2\pi M_x(\omega) = \int_{-\infty}^{\omega} S_x(u)du = \int_{-\infty}^{\omega} \lim_{T\to\infty} \frac{|X_T(u)|^2}{T} du. \tag{19}$$

If $M_x(\omega)$ is differentiable, then

$$2\pi \frac{dM_x(\omega)}{d\omega} = S_x(\omega). \tag{20}$$

Under these conditions

$$S_x(\omega) = \lim_{T\to\infty} \frac{|X_T(\omega)|^2}{T} \tag{21}$$

Taking the inverse Fourier transform of Equation (21) gives us

$$\mathscr{F}^{-1}\{S_x(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{|X_T(\omega)|^2}{T} e^{j\omega\tau} d\omega. \tag{22}$$

Interchanging the order of operation yields

$$\begin{aligned}
\mathscr{F}^{-1}\{S_x(\omega)\} &= \lim_{T\to\infty} \frac{1}{2\pi T} \int_{-\infty}^{\infty} X_T^*(\omega) X_T(\omega) e^{j\omega\tau} d\omega \\
&= \lim_{T\to\infty} \frac{1}{2\pi T} \int_{-\infty}^{\infty} \int_{-T/2}^{T/2} x^*(t) e^{j\omega t} dt \int_{-T/2}^{T/2} x(t) e^{-j\omega t'} dt' e^{j\omega\tau} d\omega \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^*(t) \int_{-T/2}^{T/2} x(t') \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega(t-t'+\tau)} d\omega \right] dt' dt.
\end{aligned} \tag{23}$$

The integration over ω in the above equation is equal to $\delta(t-t'+\tau)$, therefore

$$\begin{aligned}
\mathscr{F}^{-1}\{S_x(\omega)\} &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^*(t) \int_{-T/2}^{T/2} x(t') \delta(t-t'+\tau) dt' dt \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^*(t) x(t+\tau) dt
\end{aligned} \tag{24}$$

The inverse Fourier transform of $S_x(\omega)$ is called *autocorrelation function* of $x(t)$ and is denoted by $R_x(\tau)$.

To summarize

$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^*(t) x(t+\tau) dt \tag{25}$$

and

$$S_x(\omega) = \mathscr{F}\{R_x(\tau)\} \tag{26}$$

If the signal is periodic with period T_0 then,

$$R_x(\tau) = \frac{1}{T_0} \int_{-T_0} x^*(t) x(t+\tau) dt \tag{27}$$

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2 Random Signals and Noise

Basic Definitions

- Define an *experiment* with random *outcome*.
- Mapping of the outcome to a variable $\Rightarrow$ random variable.
- Mapping of the outcome to a function $\Rightarrow$ random function.

3.10

Probability (Cumulative) Distribution Function (cdf)

$$F_X(x) = \texttt{probability that } X \le x = P(X \le x) \tag{28}$$

3.11

Probability Density Function (pdf)

$$f_X(x) = \frac{dF_X(x)}{dx} \tag{29}$$

and

$$P(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(x) dx \tag{30}$$

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2.1 Statistical Averages

Mean of a Discrete RV

$$\bar{X} = \mathscr{E}[X] = \sum_{j=1}^{M} x_j P_j \quad (31)$$

Mean of a Continuous RV

$$\bar{X} = \mathscr{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx \quad (32)$$

Variance of a RV

$$\sigma_X^2 \triangleq \mathscr{E}\left\{[X - \mathscr{E}(X)]^2\right\} = \mathscr{E}[X^2] - \mathscr{E}^2[X] \quad (33)$$

3.13

Given a two random variables X and Y.

Covariance

$$\mu_{XY} = \mathscr{E}\{[X - \bar{x}][Y - \bar{Y}]\} = \mathscr{E}[XY] - \mathscr{E}[X]\mathscr{E}[Y] \quad (34)$$

Correlation Coefficient

$$\rho_{XY} = \frac{\mu_{XY}}{\sigma_X \sigma_Y} \quad (35)$$

Autocorrelation

$$R_X(\tau) = \mathscr{E}[X(t)X(t+\tau)] \quad (36)$$

3.14

2.2 Stochastic Processes

Terminology

See Figure 1

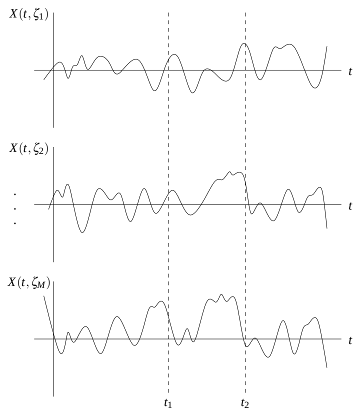

Figure 1: Sample functions of a random process

- $X(t, \zeta_i)$: sample function.
- The governing experiment: random or stochastic process.
- All sample functions: ensemble.
- $X(t_j, \zeta)$: random variable.

3.15

Strict Sense Stationarity

If the joint pdfs depend only on the time difference regardless of the time origin, then the random process is known as *stationary*.

For stationary process means and variances are independent of time and the covariance depends only on the time difference. 3.16

Wide Sense Stationarity

If the joint pdfs depends on the time difference but the mean and variances are time-independent, then the random process is known as *wide-sense-stationary*.

3.17

Ergodicity

If the time statistics equals ensemble statistics, then the random process is known as *ergodic*.

3.18

2.3 Correlation and Power Spectral Density

Power Spectral Density

Given a sample function $X(t,\zeta_i)$ of a random process, we first obtain the power spectral density by means of the Fourier transform of a truncated version $X_T(t,\zeta_i)$ defined as

$$X_T(t,\zeta_i) = \begin{cases} X(t,\zeta_i), & |t| < \frac{1}{2}T \\ 0, & \texttt{otherwise} \end{cases} \tag{37}$$

The Fourier transform of $X_T(t,\zeta_i)$ is

$$\mathscr{F}\{X_T(t,\zeta_i)\} = \int_{-T/2}^{T/2} X(t,\zeta_i) e^{j2\pi ft} dt \tag{38}$$

3.19

Power Spectral Density of a Random Process

The energy spectral density is $|\mathscr{F}\{X_T(t,\zeta_i)\}|^2$ and the average power density over the T is $|\mathscr{F}\{X_T(t,\zeta_i)\}|^2/T$. Since we have many sample functions, it is intuitive to take the ensemble average as $T \to \infty$, therefor the power spectral density, $S_X(f)$ is given by

$$S_X(f) = \lim_{T\to\infty} \frac{\overline{|\mathscr{F}\{X_T(t,\zeta_i)\}|^2}}{T} \tag{39}$$

3.20

Wiener-Khinchine Theorem

$$\begin{aligned} |\mathscr{F}[X_{2T}(t)]|^2 &= \left| \int_{-T}^{T} X(t) e^{-j\omega t} dt \right|^2 \\ &= \int_{-T}^{T} \int_{-T}^{T} X(t) X(\sigma) e^{j\omega(t-\sigma)} dt d\sigma \end{aligned} \tag{40}$$

$$\begin{aligned} \mathscr{E}\left\{|\mathscr{F}[X_{2T}(t)]|^2\right\} &= \int_{-T}^{T} \int_{-T}^{T} \mathscr{E}[X(t)X(\sigma)] e^{j\omega(t-\sigma)} dt d\sigma \\ & \int_{-T}^{T} \int_{-T}^{T} R(t-\sigma) e^{j\omega(t-\sigma)} dt d\sigma \end{aligned} \tag{41}$$

Apply the change of variables $u = t - \sigma$ and $v = t$, thus (refer to Figure 2).

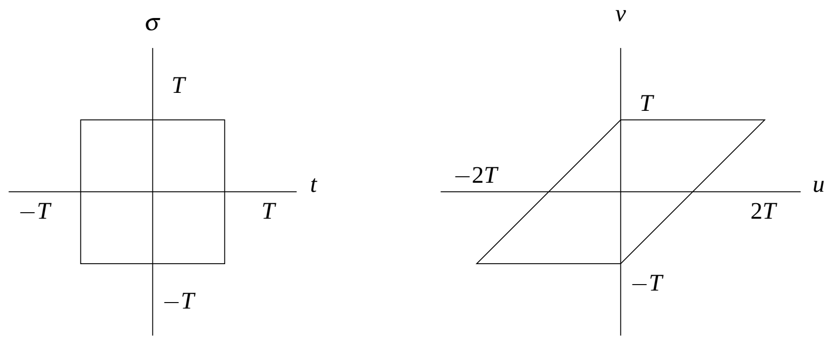

Figure 2: Sample functions of a random process

$$\begin{aligned}
&\mathscr{E}\left\{|\mathscr{F}[X_{2T}(t)]|^2\right\} \\
&\quad = \int_{u=-2T}^{0} R(u)e^{-j\omega u}du \left(\int_{-T}^{u+T} dv\right) du + \int_{u=0}^{2T} R(u)e^{-j\omega u} \left(\int_{u-T}^{T} dv\right) du \\
&\quad = \int_{-2T}^{0} (2T+u)R(u)e^{-j\omega u}du + \int_{0}^{2T} (2T-u)R(u)e^{-j\omega u}du \\
&\quad = 2T \int_{-2T}^{2T} \left(1-\frac{|u|}{2T}\right) R(u)e^{-j\omega u}du
\end{aligned} \tag{42}$$

$$S_X(f) = \lim_{T\to\infty} \int_{-2T}^{2T} \left(1-\frac{|u|}{2T}\right) R_X(u)e^{-j\omega u}du \tag{43}$$

as $T \to \infty$

$$S(f) \overset{\mathscr{F}}{\longleftrightarrow} R(\tau) \tag{44}$$

3.21

2.4 Input-Output Relationship of Linear Systems

$$S_Y(f) = |H(f)|^2 S_X(f) \tag{45}$$

3.22

3 Examples

Example 1 - Mean and Variance

Given a random variable described by the following uniform pdf

$$f_X(x) = \begin{cases} \frac{1}{b-1}, & a \le x \le b \\ 0, & \texttt{otherwise} \end{cases} \tag{46}$$

Compute the mean, the second moment, and the variance. 3.23

Example 2 - Time and Statistical Averages

Given a the following random process

$$X(t) = A\cos(2\pi f_0 t + \Theta) \tag{47}$$

where f_0 is a constant and Θ is a random variable with the following pdf

$$f_\Theta(x) = \begin{cases} \frac{1}{2\pi}, & |\theta| \le \pi \\ 0, & \texttt{otherwise.} \end{cases} \tag{48}$$

Compute the statistical and time averages of the first and second moments. Is this process stationary? Is it ergodic? 3.24

Example 3 - Power Spectral Density

Given the same process shown in Example 2, compute the power spectral density using Eq. 39. Verify your answer using Wiener-Khinchine Theorem. 3.25