EE 521: Instrumentation and Measurements

Aly El-Osery

Electrical Engineering Department, New Mexico Tech
Socorro, New Mexico, USA

October 7, 2009

1. **Review Material**
   - Signal Classification
   - Time Averages
   - Frequency Domain

2. **Random Signals and Noise**
   - Statistical Averages
   - Stochastic Processes
   - Correlation and Power Spectral Density
   - Input-Output Relationship of Linear Systems

3. **Examples**

Assume the voltage across a resistor R is $e(t)$ and is producing a current $i(t)$. The instantaneous power per ohm is $p(t) = e(t)i(t)/R = i^2(t)$.

Total Energy

$$E = \lim_{T\to\infty} \int_{-T}^{T} i^2(t)dt \tag{1}$$

Average Power

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} i^2(t)dt \tag{2}$$

Arbitrary signal $x(t)$

Total Normalized Energy

$$E \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt \tag{3}$$

Normalized Power

$$P \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \tag{4}$$

For Energy Signals

$$\phi(\tau) = \int_{-\infty}^{\infty} x(\lambda)x(\lambda + \tau)df \tag{5}$$

For Power Signals

$$R(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t + \tau)dt \tag{6}$$

For Periodic Signals

$$R(\tau) = \frac{1}{T_0} \int_{T_0} x(t)x(t + \tau)dt \tag{7}$$

Fourier Transform Equations

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \tag{8}$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \tag{9}$$

Energy Spectral Density

Rayleigh's Energy Theorem or Parseval's theorem

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df \qquad (10)$$

Energy Spectral Density

$$G(f) \triangleq |X(f)|^2 \qquad (11)$$

with units of *volts*^{2}-*sec*^{2} or, if considered on a per-ohm basis, *watts-sec/Hz=joules/Hz*

Power Spectral Density

$$P = \int_{-\infty}^{\infty} S(f)df = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \qquad (12)$$

where we define $S(f)$ as the power spectral density with units of watts/Hz.

Basic Definitions

- Define an *experiment* with random *outcome*.
- Mapping of the outcome to a variable $\Rightarrow$ random variable.
- Mapping of the outcome to a function $\Rightarrow$ random function.

Probability (Cumulative) Distribution Function (cdf)

$$F_X(x) = \texttt{probability that } X \leq x = P(X \leq x) \tag{13}$$

Probability Density Function (pdf)

$$f_X(x) = \frac{dF_X(x)}{dx} \tag{14}$$

and

$$P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(x)dx \tag{15}$$

Mean of a Discrete RV

$$\bar{X} = \mathcal{E}[X] = \sum_{j=1}^{M} x_j P_j \qquad (16)$$

Mean of a Continuous RV

$$\bar{X} = \mathcal{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx \qquad (17)$$

Variance of a RV

$$\sigma_X^2 \triangleq \mathcal{E}\left\{[X - \mathcal{E}(X)]^2\right\} = \mathcal{E}[X^2] - \mathcal{E}^2[X] \qquad (18)$$

Given a two random variables X and Y.

Covariance

$$\mu_{XY} = \mathcal{E}\left\{[X - \bar{x}][Y - \bar{Y}]\right\} = \mathcal{E}[XY] - \mathcal{E}[X]\mathcal{E}[Y] \tag{19}$$

Correlation Coefficient

$$\rho_{XY} = \frac{\mu_{XY}}{\sigma_X \sigma_Y} \tag{20}$$

Autocorrelation

$$R_X(\tau) = \mathcal{E}[X(t)X(t+\tau)] \tag{21}$$

Terminology

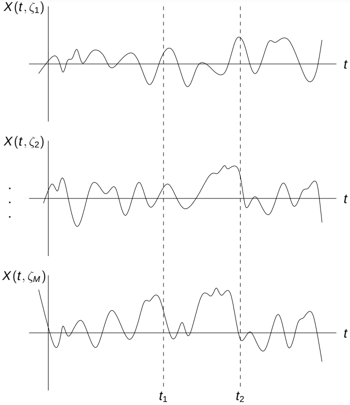

Figure: Sample functions of a random process

- $X(t, \zeta_i)$: sample function.
- The governing experiment: random or stochastic process.
- All sample functions: ensemble.
- $X(t_j, \zeta)$: random variable.

Strict Sense Stationarity

If the joint pdfs depend only on the time difference regardless of the time origin, then the random process is known as *stationary*.

For stationary process means and variances are independent of time and the covariance depends only on the time difference.

Wide Sense Stationarity

If the joint pdfs depends on the time difference but the mean and variances are time-independent, then the random process is known as *wide-sense-stationary*.

Ergodicity

If the time statistics equals ensemble statistics, then the random process is known as *ergodic*.

Power Spectral Density

Given a sample function $X(t, \zeta_i)$ of a random process, we first obtain the power spectral density by means of the Fourier transform of a truncated version $X_T(t, \zeta_i)$ defined as

$$X_T(t, \zeta_i) = \begin{cases} X(t, \zeta_i), & |t| < \frac{1}{2}T \\ 0, & \texttt{otherwise} \end{cases} \tag{22}$$

The Fourier transform of $X_T(t, \zeta_i)$ is

$$\mathcal{F}\{X_T(t, \zeta_i)\} = \int_{-T/2}^{T/2} X(t, \zeta_i) e^{j2\pi ft} dt \tag{23}$$

Power Spectral Density of a Random Process

The energy spectral density is $|\mathcal{F}\{X_T(t,\zeta_i)\}|^2$ and the average power density over the T is $|\mathcal{F}\{X_T(t,\zeta_i)\}|^2/T$. Since we have many sample functions, it is intuitive to take the ensemble average as $T \to \infty$, therefor the power spectral density, $S_X(f)$ is given by

$$S_X(f) = \lim_{T\to\infty} \frac{\overline{|\mathcal{F}\{X_T(t,\zeta_i)\}|^2}}{T} \tag{24}$$

Wiener-Khinchine Theorem

$$S_X(f) = \lim_{T\to\infty} \int_{-2T}^{2T} \left(1 - \frac{|u|}{2T}\right) R_X(u) e^{-j\omega u} du \tag{25}$$

as $T \to \infty$

$$S(f) \overset{\mathcal{F}}{\longleftrightarrow} R(\tau) \tag{26}$$

$$S_Y(f) = |H(f)|^2 S_X(f) \tag{27}$$

Example 1 - Mean and Variance

Given a random variable described by the following uniform pdf

$$f_X(x) = \begin{cases} \frac{1}{b-1}, & a \le x \le b \\ 0, & \texttt{otherwise} \end{cases} \tag{28}$$

Compute the mean, the second moment, and the variance.

Example 2 - Time and Statistical Averages

Given a the following random process

$$X(t) = A\cos(2\pi f_0 t + \Theta) \qquad (29)$$

where f_0 is a constant and Θ is a random variable with the following pdf

$$f_\Theta(x) = \begin{cases} \frac{1}{2\pi}, & |\theta| \leq \pi \\ 0, & \texttt{otherwise}. \end{cases} \qquad (30)$$

Compute the statistical and time averages of the first and second moments. Is this process stationary? Is it ergodic?

Example 3 - Power Spectral Density

Given the same process shown in Example 2, compute the power spectral density using Eq. 24. Verify your answer using Wiener-Khinchine Theorem.