Lecture 7

Discrete Systems

EE 521: Instrumentation and Measurements

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1 The z-Transform

Definition

The z-transform of a discrete-time signal $x(n)$ is defined as the power series

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{1}$$

where z is a complex variable, and hence, can be represented as a magnitude and phase.

$$z = re^{j\theta} \tag{2}$$

It exists only for those values of z for which this series converges. 7.3

Summation Formula

$$\sum_{n=M}^{N} a^n = \begin{cases} \frac{a^M - a^{N+1}}{1-a}, & \text{if} \quad a \neq 1 \\ N - M + 1, & \text{if} \quad a = 1 \end{cases} \tag{3}$$

7.4

Right-Sided Sequence

$$x(n) = a^n u(n) \quad \stackrel{\mathscr{Z}}{\longleftrightarrow} \quad X(z) = \frac{1}{1 - az^{-1}} \tag{4}$$

This definition is not complete without stating the region-of-convergence (ROC). See Figure 2. 7.5

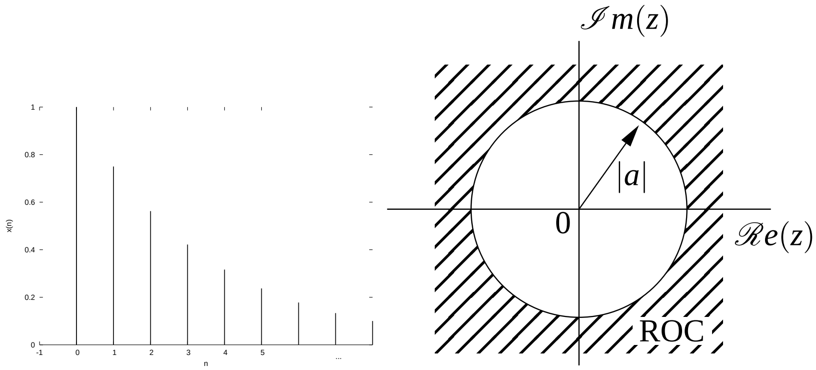

Figure 1: Region-of-convergence of right-sided sequence

left-Sided Sequence

$$x(n) = -a^n u(-n-1) \quad \overset{\mathscr{Z}}{\longleftrightarrow} \quad X(z) = \frac{1}{1-az^{-1}} \tag{5}$$

This definition is not complete without stating the region-of-convergence (ROC). See Figure 2.

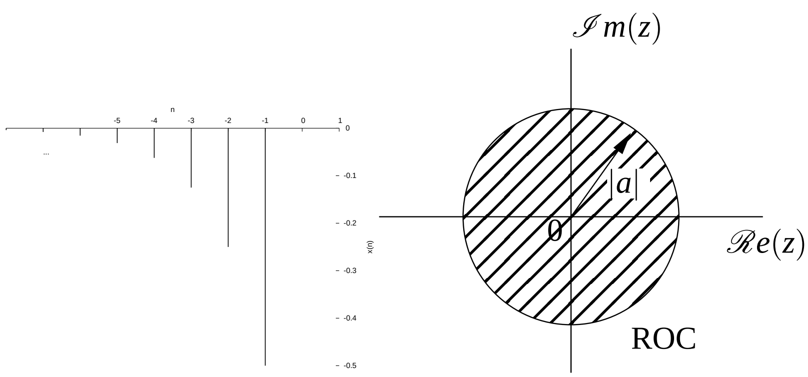

Figure 2: Region-of-convergence of left-sided sequence

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Poles and Zeros

If $X(z)$ is rational function, then

$$\begin{aligned} X(z) = \frac{B(z)}{A(z)} &= \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}} \\ &= \frac{b_0}{a_0} z^{-M+N} \frac{(z-z_1)(z-z_2)\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)} \end{aligned} \tag{6}$$

$z = z_1, z_2, \ldots, z_M$ and $p = p_1, p_2, \ldots, p_N$ are the zeros and poles of $X(z)$, respectively. 7.7

2 Linear Time-Invariant System

A relaxed linear time-invariant system with input $x(n)$, an impulse response $h(n)$, and an output $y(n)$ can be expressed in the z-domain as

$$Y(z) = H(z)X(z) \tag{7}$$

where H(z) is the transfer function of the system. 7.8

Difference Equation

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \tag{8}$$

and hence,

$$\frac{Y(z)}{X(z)} = H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \tag{9}$$

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Causality and Stability

Causality
A system is causal if the ROC is the exterior of a circle.

Stability
A system is stable if the ROC includes the unit circle.

Causal and Stable
A system is both causal and stable if the poles are inside the unit circle. 7.10

Response to Complex Exponential

Using the convolution equation

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) \tag{10}$$

If this system is excited by a complex exponential

$$x(n) = Ae^{j\omega n}, \qquad -\infty < n < \infty \tag{11}$$

where ω is an arbitrary frequency, then

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} h(k)Ae^{j\omega(n-k)} \\ &= A\left[\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}\right]e^{j\omega n} \\ &= AH(\omega)e^{j\omega n} \end{aligned} \tag{12}$$

Assuming of course that the system is stable so that

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} \tag{13}$$

exists. 7.11

Relationship to Fourier Transform

If the ROC includes the unit circle, then

$$H(\omega) = H(z)|_{z=e^{j\omega}} = \sum_{k=-\infty}^{\infty} h(n)e^{j\omega n} \tag{14}$$

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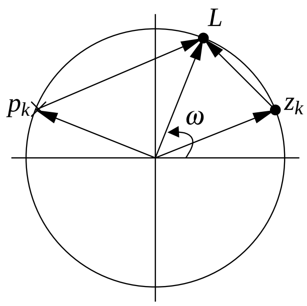

Figure 3: Frequency response using the location of poles and zeros

Effect of Pole-Zero Location

See Figure 3.

$$mag = \frac{product\ of\ distances\ from\ all\ zeros\ to\ L}{product\ of\ distances\ from\ all\ poles\ to\ L} \tag{15}$$

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3 Filter Design

Filter Specs

See Figure 4.

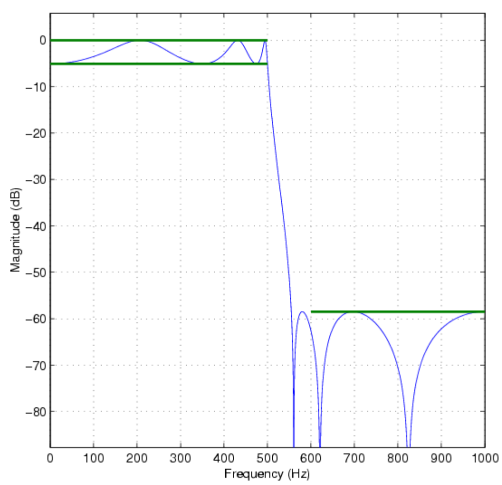

Figure 4: Filter Specifications

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3.1 IIR Filters

By Pole-Zero Placement

- Place zeros at locations you want the magnitude to decrease.
- Place Poles at locations you want the magnitude to increase.

This method has the advantage of being simple and doesn't require a lot of tools. The disadvantage is that it doesn't provide much control in the specs of the filter. 7.15

Transformation from Analog Filters

Use a transformation to convert analog filters to digital filters. The bilinear transformation is frequently used.

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \tag{16}$$

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Transformation from Analog Filters

The bilinear transformation causes a frequency warping described by

$$\omega = 2\tan^{-1}\frac{\Omega T}{2} \tag{17}$$

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Transformation from Analog Filters

Some common analog filter includes

- Butterworth: No ripple in either passband or stopband.
- Type I Chebyshev: ripple in passband only.
- Type II Chebyshev: ripple in stopband only.
- Elliptic: ripple in both passband and stopband. 7.18

Quantization Effects

Due to quantization, the location of poles and zeros will be different than designed. This will cause the frequency response to be modified. To minimize the effects of quantization, build your filter as a cascade of second order systems. 7.19

3.2 FIR Filters

A system using FIR filters has the output characterized by

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k) \tag{18}$$

FIR filters with symmetric and antisymmetric impulse responses have linear phase. 7.20

FIR Filter Design Using Windows

1. Compute the infinite impulse response from the an ideal impulse response.

$$h_d(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}d\omega \tag{19}$$

2. Compute a finite impulse response using a finite length window.

$$h(n) = h_d(n)w(n) \tag{20}$$

3. Windows include rectangular, Hamming, Hanning, Bartlet, Blackman, etc.
4. The wider the window, the sharper the cut-off.

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See Figure 5

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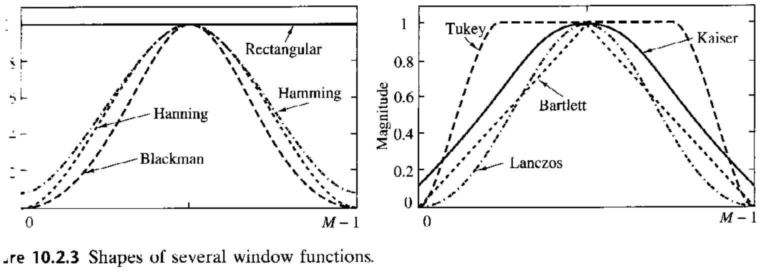

Figure 5: ©Proakis and Manolakis, *Digital Signal Processing*, 4th Edition, Prentice Hall, 2007

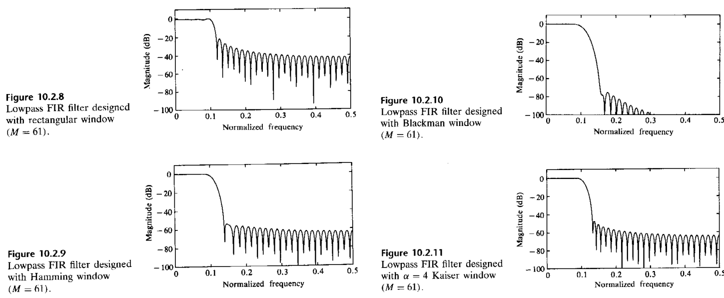

Figure 6: ©Proakis and Manolakis, *Digital Signal Processing*, 4th Edition, Prentice Hall, 2007

See Figure 6

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FIR Filter Design Using Frequency-Sampling Method

1. Design the required filter in the Frequency domain.
2. Sample the designed magnitude response.
3. Add the linear phase.
4. Compute the IDFT to obtain $h(n)$.

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See Figure 7

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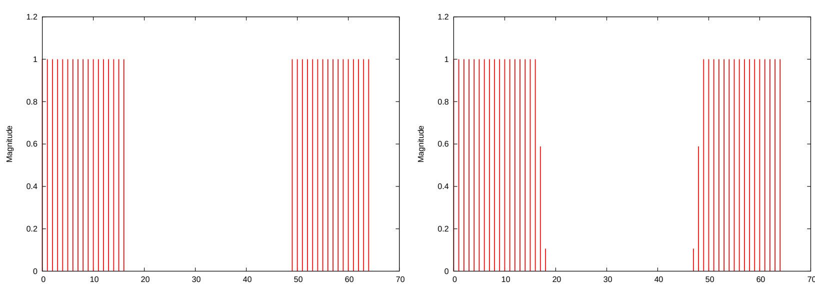

Figure 7: Frequency sampling approach with not transition points (left) and with transition points (right).

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See Figure 8

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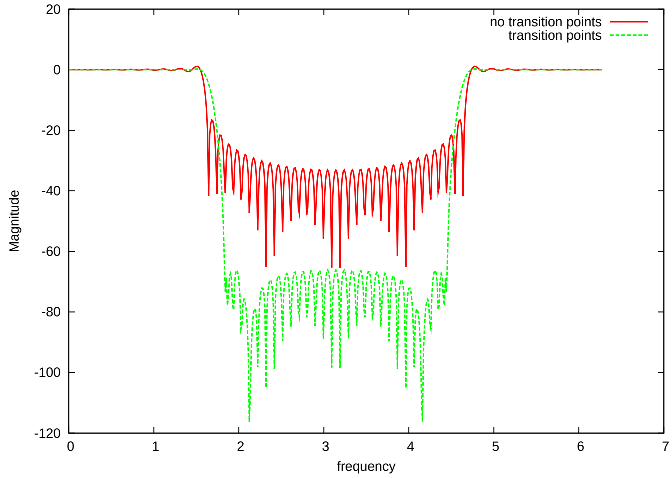

Figure 8: Magnitude response using frequency sampling approach with not transition points (left) and with transition points (right).

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FIR Filter Design Using Equiripple Method

Uses the Alternation Theorem and solves an optimization problem resulting in an equal ripple. Design is accomplished by specifying bandedges and a weighting function. 7.27