EE 521: Instrumentation and Measurements

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1. Signal is stochastic due to noise.
2. True spectral density is not known.
3. Signal is sampled and time limited resulting in loss of resolution, frequency domain aliasing and spectral leakage.
4. Only one sample function may be available.

True Autocorrelation and Power Spectral Density

If $x(t)$ is stationary random process then,

Ensemble Autocorrelation

$$\gamma_{xx} = \mathcal{E}[x^*(t)x(t+\tau)] \tag{1}$$

Power Spectral Density

$$\Gamma_{xx}(F) = \int_{-\infty}^{\infty} \gamma_{xx} e^{-j2\pi Ft} dt \tag{2}$$

Time-averages

Using a single sample function,

Time Autocorrelation

$$R_{XX}(\tau) = \frac{1}{2T_0} \int_{-T_0}^{T_0} x^*(t)x(t+\tau)dt \tag{3}$$

If the process is ergodic, then we can use the time-average autocorrelation $R_{XX}(\tau)$, then

$$\begin{aligned} \gamma_{XX}(\tau) &= \lim_{T_0 \to \infty} R_{XX}(\tau) \\ &= \lim_{T_0 \to \infty} \frac{1}{2T_0} \int_{-T_0}^{T_0} x^*(t)x(t+\tau)dt \end{aligned} \tag{4}$$

Overview

We can compute an estimate $P_{XX}(F)$ of the true power spectral density $\Gamma_{XX}(F)$ by

Direct Method

$$\begin{aligned} P_{XX}(F) &= \int_{-T_0}^{T_0} R_{XX}(\tau) e^{-j2\pi F\tau} d\tau \\ &= \frac{1}{2T_0} \left| \int_{-T_0}^{T_0} x(t) e^{-j2\pi Ft} dt \right|^2 \end{aligned} \quad (5)$$

Indirect Method

Compute the autocorrelation function $R_{XX}(\tau)$ then compute the Fourier transform.

Overview

The actual power spectral density is computed as

$$\begin{aligned} \Gamma_{XX}(\tau) &= \lim_{T \to \infty} \mathcal{E}[P_{XX}(F)] \\ &= \lim_{T \to \infty} \mathcal{E}\left[\frac{1}{2T_0} \left| \int_{-T_0}^{T_0} x(t) e^{-j2\pi F t} dt \right|^2 \right] \end{aligned} \qquad (6)$$

Sampled Signals

In practice we need to sample the continuous time signal in order to estimate its power spectral density. Using a finite duration sampled signal,

$$r'_{XX}(m) = \frac{1}{N-M} \sum_{n=0}^{N-m-1} x^*(n)x(n+m) \quad (7)$$

and its Fourier transform is

$$P'_{XX}(f) = \sum_{m=-N+1}^{N+1} r'_{XX}(m)e^{-j2\pi fm} \quad (8)$$

Unbiased Estimate

$$\mathcal{E}[r'_{xx}(m)] = \frac{1}{N-m} \sum_{n=0}^{N-m-1} \mathcal{E}[x^*(n)x(n+m)] \quad (9)$$
$$= \gamma_{xx}(m)$$

This is known as *unbiased estimate* estimate of the true (statistical) autocorrelation of $x(n)$

Consistent Estimate

$$var[r'_{xx}(m)] \approx \frac{N}{[N-m]^2} \sum_{n=-\infty}^{\infty} \left[|\gamma_{xx}(n)|^2 + \gamma^*_{xx}(n-m)\gamma_{xx}(n+m) \right] \tag{10}$$

$$\lim_{N\to\infty} var[r'_{xx}(m)] = 0 \tag{11}$$

This is known as *consistent* estimate of the true autocorrelation of $x(n)$.

An Alternative Way

For large lags the previous method has large variance. An alternative way is to compute the estimate using

$$r_{XX}(m) = \frac{1}{N} \sum_{n=0}^{N-m-1} x^*(n)x(n+m), \quad 0 \leq m \leq N-1 \qquad (12)$$

$$r_{XX}(m) = \frac{1}{N} \sum_{n=|m|}^{N-m-1} x^*(n)x(n+m), \quad -1 \leq m \leq 1-N \qquad (13)$$

An Alternative Way

This method has a biased given by

$$\mathcal{E}[r_{XX}(m)] = \left(1 - \frac{|m|}{N}\right)\gamma_{XX}(m) \tag{14}$$

which approaches $\gamma_{XX}(m)$ as N goes to ∞. This is known as *asymptotically unbiased*. The advantage is that the variance is smaller.

Using the Direct Method

$$
\begin{aligned}
P_{XX}(f) &= \sum_{m=-(N-1)}^{N-1} r_{XX}(m) e^{-j2\pi fm} \\
&= \frac{1}{N} \left| \sum_{n=0}^{N-1} x(n) e^{-j2\pi fn} \right|^2 = \frac{1}{N} |X(f)|^2
\end{aligned} \tag{15}
$$

and

$$
\mathcal{E}[P_{XX}(f)] = \sum_{m=-(N-1)}^{N-1} \left(1 - \frac{|m|}{N}\right) \gamma_{XX}(m) e^{-j2\pi fm} \tag{16}
$$

which is a windowed version of γ_{XX}, i.e.,

$$
\tilde{\gamma}_{XX} = \left(1 - \frac{|m|}{N}\right) \gamma_{XX}(m) \tag{17}
$$

Remarks on Periodogram

- Asymptotically unbiased.
- Variance does not approach 0 as $N \to \infty$, and therefore, periodograms are not a consistent estimate of the true power density spectrum.

Bartlett Method

In order to reduce the variance.

- Subdivide the sequence into smaller nonoverlappling segments.
- Compute the periodogram of each segment.
- Average the perodograms.
- Frequency resolution

$$\Delta f = \frac{0.9}{M} \tag{18}$$

where M is the number of segments

Welch Method

- A modified version of the Bartlett method by allowing overlap of segments.
- Window the data segments before computing the periodogram.
- Frequency resolution

$$\Delta f = \frac{1.28}{M} \tag{19}$$

Blackman and Tukey Method

- Window the autocorrelation sequence to reduce the effect of the unreliable values at large lags.
- Fourier transform the windowed autocorrelation.
- Frequency resolution

$$\Delta f = \frac{0.64}{M} \qquad (20)$$

where $2M + 1$ is the window size.

Quality Measure

$$Q_A = \frac{\{\mathcal{E}[P_{XX}(f)]\}^2}{var[P_{XX}(f)]} \tag{21}$$

Estimate	*Quality Factor*	*Number of Computation*
Bartlett	$1.11N\Delta f$	$\frac{N}{2}\log_2\frac{0.9}{\Delta f}$
Welch (50% overlap)	$1.39N\Delta f$	$N\log_2\frac{5.12}{\Delta f}$
Blackman-Tukey	$2.34N\Delta f$	$N\log_2\frac{1.28}{\Delta f}$

Sequence Modeling

Model the data sequence $x(n)$ as the output of a linear system characterized by

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{q} b_k z^{-k}}{1 + \sum_{k=1}^{p} a_k z^{-k}} \tag{22}$$

and the corresponding difference equation

$$x(n) = -\sum_{k=1}^{p} a_k x(n-k) + \sum_{k=0}^{q} b_k w(n-k) \tag{23}$$

where $w(n)$ is zero-mean white noise.

Model Types

1. Pole-zero model is known as *autoregressive moving average* (ARMA).
2. Only poles model is known as *autoregressive* (AR).
3. Only zeros model is known as *moving average* (MA).

AR models are very commonly used as they are less order than MA, they are simple to represent, and they can represent narrow peaks.