EE 451: Digital Signal Processing

Stochastic Processes and Spectral Estimation

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November 29, 2011

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The signal is stochastic in nature.

Assume the voltage across a resistor R is $e(t)$ and is producing a current $i(t)$. The instantaneous power per ohm is $p(t) = e(t)i(t)/R = i^2(t)$.

Total Energy

$$E = \lim_{T\to\infty} \int_{-T}^{T} i^2(t)dt \tag{1}$$

Average Power

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} i^2(t)dt \tag{2}$$

Arbitrary signal $x(t)$

Total Normalized Energy

$$E \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt \tag{3}$$

Normalized Power

$$P \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \tag{4}$$

For Energy Signals

$$\phi(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)dt \tag{5}$$

For Power Signals

$$R(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t+\tau)dt \tag{6}$$

For Periodic Signals

$$R(\tau) = \frac{1}{T_0} \int_{T_0} x(t)x(t+\tau)dt \tag{7}$$

Energy Spectral Density

Rayleigh's Energy Theorem or Parseval's theorem

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(F)|^2 dF \tag{8}$$

Energy Spectral Density

$$G(F) \triangleq |X(F)|^2 \tag{9}$$

with units of *volts*^{2}-*sec*^{2} or, if considered on a per-ohm basis, *watts-sec/Hz=joules/Hz*

Power Spectral Density

$$P = \int_{-\infty}^{\infty} S(F)dF = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \tag{10}$$

where we define $S(F)$ as the power spectral density with units of watts/Hz.

Basic Definitions

- Define an *experiment* with random *outcome*.
- Mapping of the outcome to a variable $\Rightarrow$ random variable.
- Mapping of the outcome to a function $\Rightarrow$ random function.

Probability (Cumulative) Distribution Function (cdf)

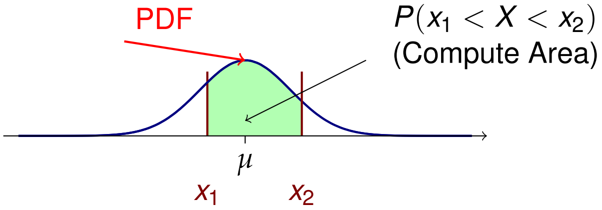
$$F_X(x) = \texttt{probability that } X \leq x = P(X \leq x) \tag{11}$$

Probability Density Function (pdf)

$$f_X(x) = \frac{dF_X(x)}{dx} \qquad (12)$$

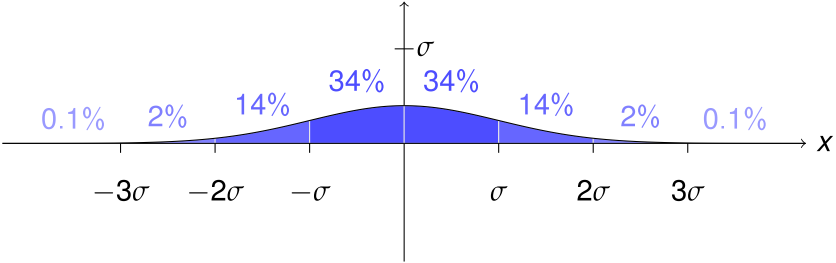
and

$$P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(x)dx \qquad (13)$$

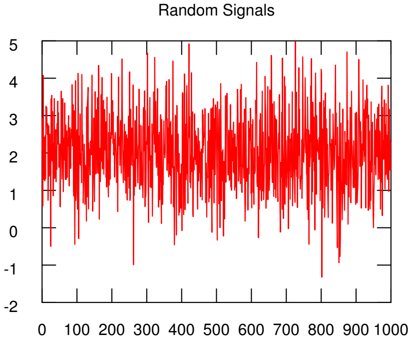


Gaussian Distribution

$$\frac{1}{\sigma\sqrt{2\pi}}\exp\left(\frac{-x^2}{2\sigma^2}\right)$$



PDF of White Noise



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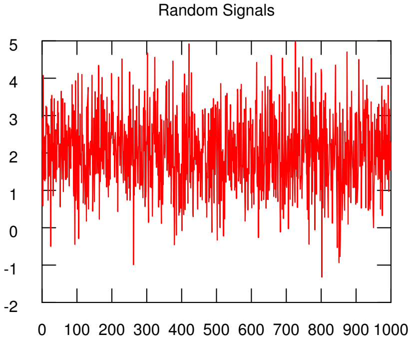

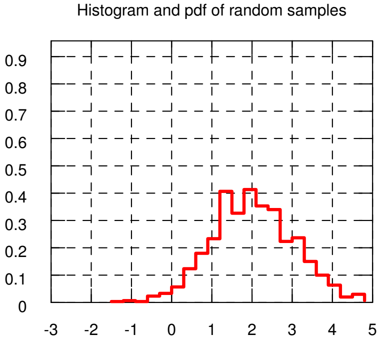

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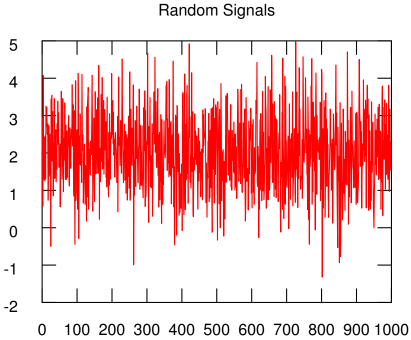

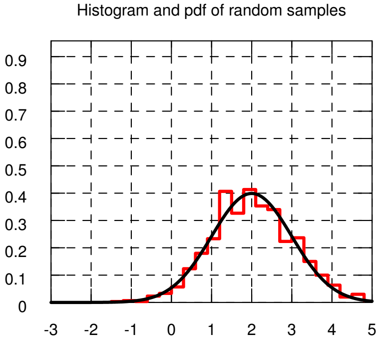

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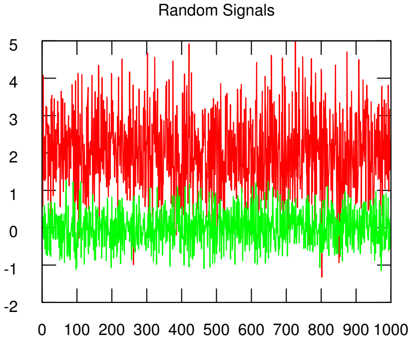

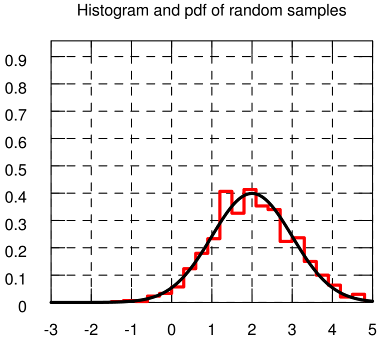

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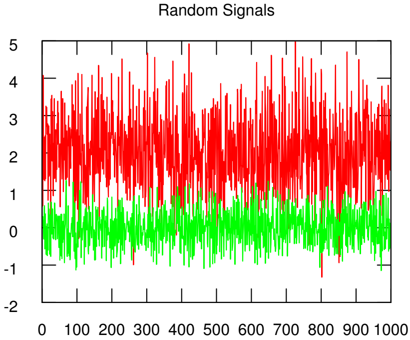

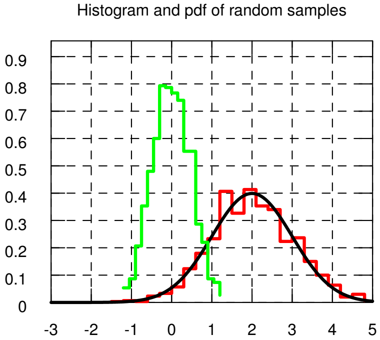

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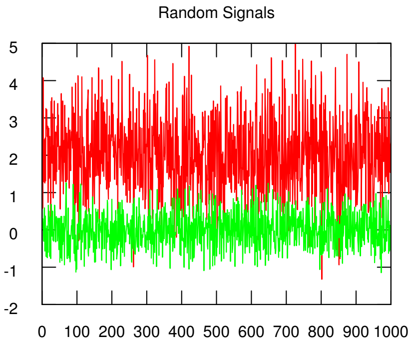
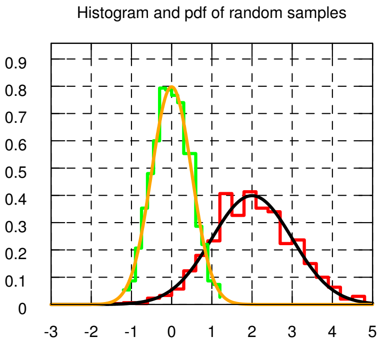




Mean of a Discrete RV

$$\bar{X} = \mathbb{E}[X] = \sum_{j=1}^{M} x_j P_j \tag{14}$$

Mean of a Continuous RV

$$\bar{X} = \mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx \tag{15}$$

Variance of a RV

$$\sigma_X^2 \triangleq \mathbb{E}\left\{[X - \mathbb{E}(X)]^2\right\} = \mathbb{E}[X^2] - \mathbb{E}^2[X] \tag{16}$$

Given a two random variables X and Y.

Covariance

$$\mu_{XY} = \mathbb{E}\left\{[X - \bar{x}][Y - \bar{Y}]\right\} = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \tag{17}$$

Correlation Coefficient

$$\rho_{XY} = \frac{\mu_{XY}}{\sigma_X \sigma_Y} \tag{18}$$

Autocorrelation

$$\Gamma_X(\tau) = \mathbb{E}[X(t)X(t+\tau)] \tag{19}$$

Terminology

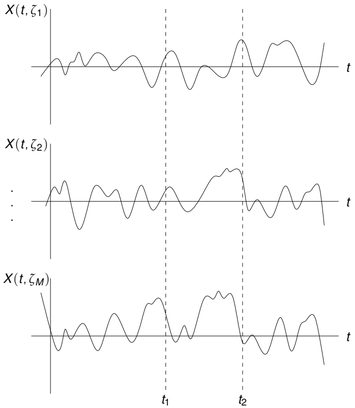


Figure: Sample functions of a random process

- $X(t, \zeta_i)$: sample function.
- The governing experiment: random or stochastic process.
- All sample functions: ensemble.
- $X(t_j, \zeta)$: random variable.

Strict Sense Stationarity

If the joint pdfs depend only on the time difference regardless of the time origin, then the random process is known as *stationary*.

For stationary process means and variances are independent of time and the covariance depends only on the time difference.

Wide Sense Stationarity

If the joint pdfs depends on the time difference but the mean and variances are time-independent, then the random process is known as *wide-sense-stationary*.

Ergodicity

If the time statistics equals ensemble statistics, then the random process is known as *ergodic*.

Power Spectral Density

Given a sample function $X(t,\zeta_i)$ of a random process, we first obtain the power spectral density by means of the Fourier transform of a truncated version $X_T(t,\zeta_i)$ defined as

$$X_T(t,\zeta_i) = \begin{cases} X(t,\zeta_i), & |t| < \frac{1}{2}T \\ 0, & \texttt{otherwise} \end{cases} \tag{20}$$

The Fourier transform of $X_T(t,\zeta_i)$ is

$$\mathcal{F}\{X_T(t,\zeta_i)\} = \int_{-T/2}^{T/2} X(t,\zeta_i) e^{j2\pi Ft} dt \tag{21}$$

Power Spectral Density of a Random Process

The energy spectral density is $|\mathcal{F}\{X_T(t,\zeta_i)\}|^2$ and the average power density over the T is $|\mathcal{F}\{X_T(t,\zeta_i)\}|^2/T$. Since we have many sample functions, it is intuitive to take the ensemble average as $T \rightarrow \infty$, therefor the power spectral density, $S_X(F)$ is given by

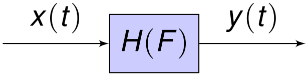
$$S_X(F) = \lim_{T\rightarrow\infty} \frac{\overline{|\mathcal{F}\{X_T(t,\zeta_i)\}|^2}}{T} \tag{22}$$

Wiener-Khinchine Theorem

$$S_X(F) = \lim_{T\to\infty} \int_{-2T}^{2T} \left(1 - \frac{|u|}{2T}\right) \Gamma_X(u) e^{-j\Omega u} du \tag{23}$$

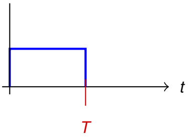
as $T \to \infty$

$$S(F) \stackrel{\mathcal{F}}{\longleftrightarrow} \Gamma(\tau) \tag{24}$$



$$S_Y(F) = |H(F)|^2 S_X(F) \tag{25}$$

Big Picture



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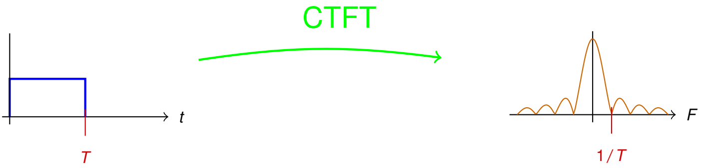

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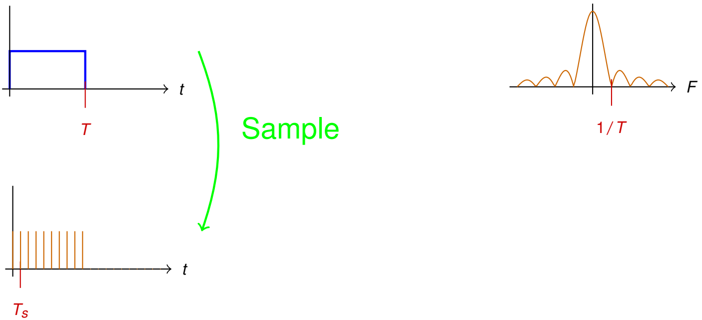

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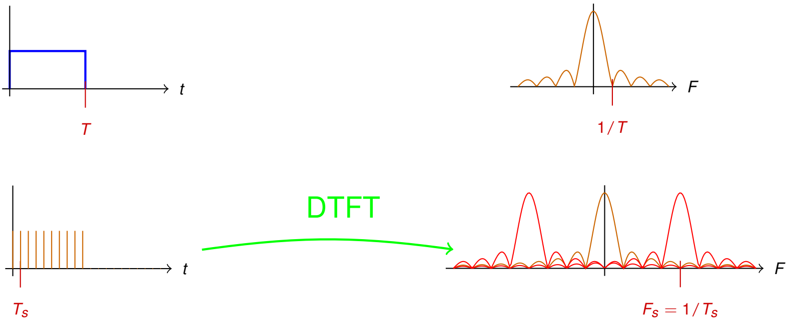

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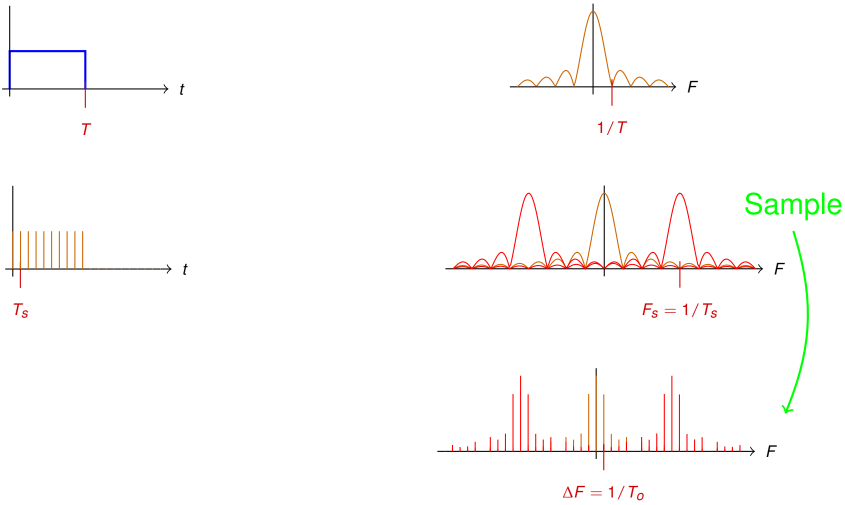

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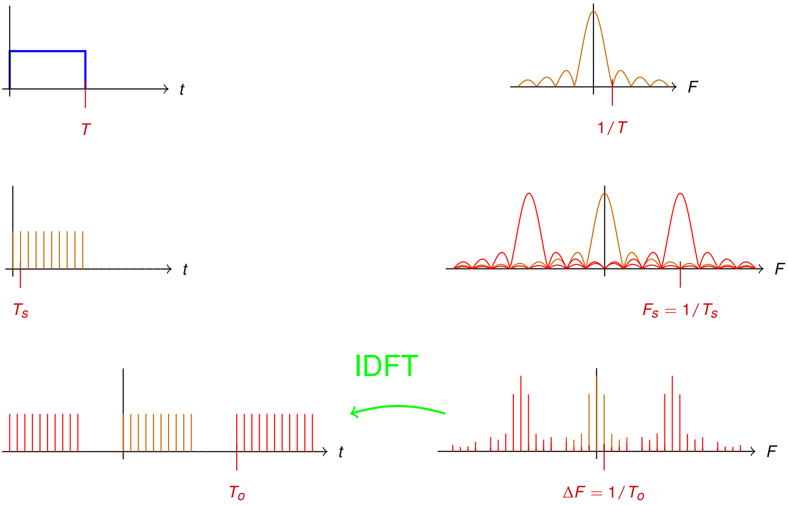

Sampling Remarks

- Must sample more than twice bandwidth to avoid aliasing.
- FFT represents a periodic version of the time domain signal $\rightarrow$ could have time domain aliasing.
- Number of points in FFT is the same as number of points in time domain signal.

Obtaining PSD for Discrete Signals

What we want is

$$\Gamma_X(\tau) = \mathbb{E}[X(t)X(t+\tau)] \xrightarrow{\mathcal{CTFT}} S_X(F)$$

For infinitely long signals.

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$$\Gamma_X(\tau) = \mathbb{E}[X(t)X(t+\tau)] \xrightarrow{\mathcal{CTFT}} S_X(F)$$

For infinitely long signals.

What we can compute is

$$\gamma_X(m) = \mathbb{E}[X(n)X(n+m)] \xrightarrow{\mathcal{DFT}} P_X(f)$$

For finite length signals.

What do we need in an estimate

As $N \to \infty$ and in the mean squared sense

Unbiased

Asymptotically the mean of the estimate approaches the true power.

Variance

Variance of the estimate approaches zero.

Resulting in a consistent estimate of the power spectrum.

Possible PSD Options

Periodogram

computed using $1/N$ times the magnitude squared of the FFT

$$\lim_{N\to\infty} \mathbb{E}[P_X(f)] = S_X(f)$$

$$\lim_{N\to\infty} var[P_X(f)] = S_X^2(f)$$

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Welch Method

computed by segmenting the data (allowing overlaps), windowing the data in each segment then computing the average of the resultant priodogram

$$\mathbb{E}[P_X(f)] = \frac{1}{2\pi MU} S_X(f) \circledast W(f)$$

$$var[P_X(f)] \approx \frac{9}{8L} S_X^2(f)$$

Welch Method

Assuming data length N, segment length M, Bartlett window, and 50% overlap

- FFT length = $M = 1.28/\Delta f = 1.28 F_s/\Delta F$
- Resulting number of segments = $L = \frac{2N}{M}$
- Length of data collected in sec. = $\frac{1.28L}{2\Delta F}$

pwelch Function

```
[Pxx,f] = pwelch(x,window,noverlap,...
                 nfft,fs,'range')
```

You can use [] in fields that you want the default to be used.

pwelch Function - WGN signal

```
Fs = 1000;
x = sqrt(0.1*Fs)*randn(1,100000);
[Pxx,f] = pwelch(x,1024,[],[],Fs,'onesided');
```

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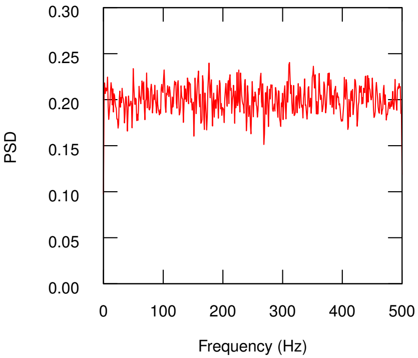

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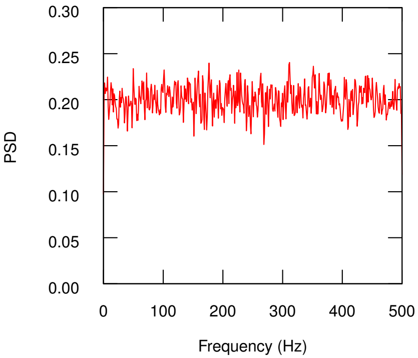

- Variance to high.

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[Pxx,f] = pwelch(x,128,[],[],Fs,'onesided')
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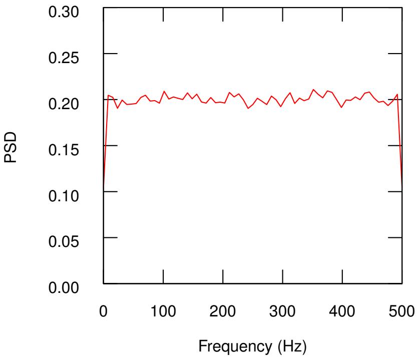

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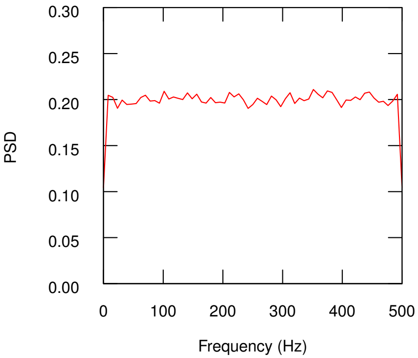

- Reduced window size.
- Variance is now smaller.

pwelch Function - cos + WGN signal

```
Fs = 100;    t = 0:1/Fs:5;
x = cos(2*pi*10*t)+cos(2*pi*11*t)+...
        sqrt(0.1*Fs)*randn(1,length(t));
[Pxx,f] = pwelch(x,1024,[],[],Fs,'onesided');
```

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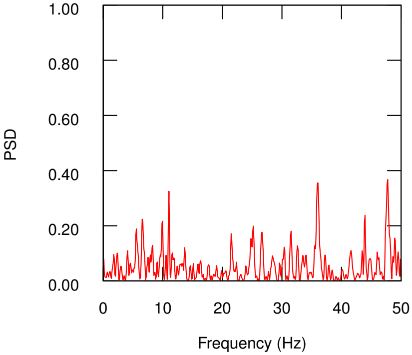

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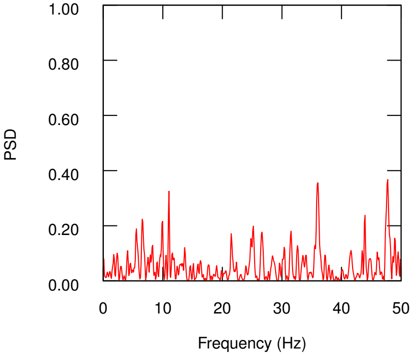

- Window larger than length of data.
- Frequency components can't be resolved.
- Variance high.

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[Pxx,f] = pwelch(x,1024,[],4096,Fs,'onesided');
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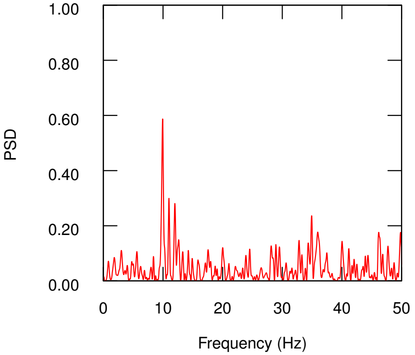

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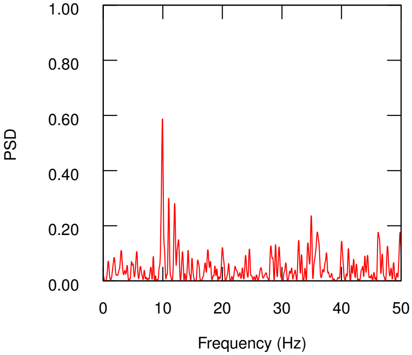

- As expected increasing `nFFT` does not help.

pwelch Function - cos + WGN signal

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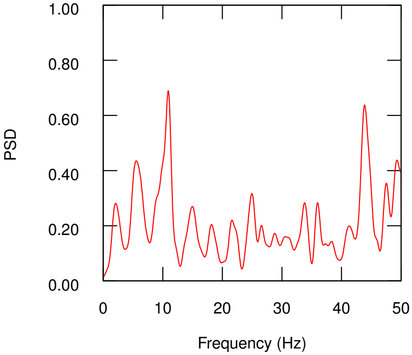

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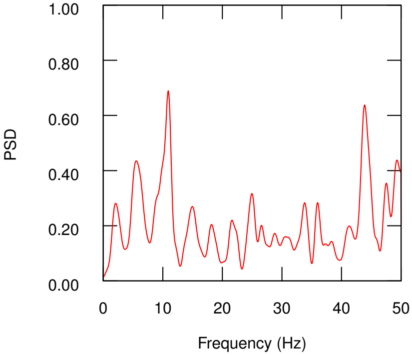

- Decreasing the window size decreases the variance.
- Still can't resolve the two frequencies.

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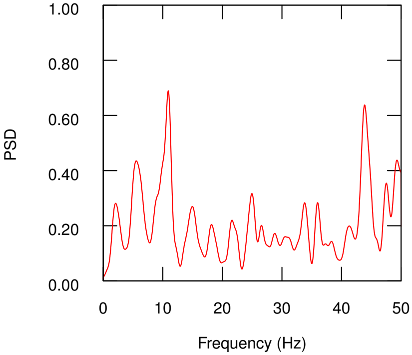

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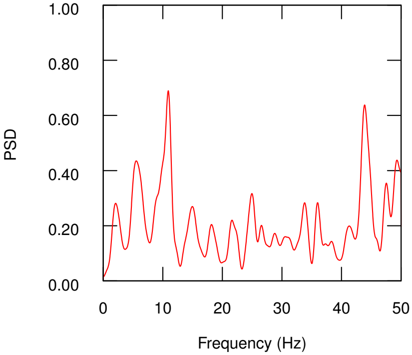
```
Fs = 100;   t = 0:1/Fs:50;
x = cos(2*pi*10*t)+cos(2*pi*11*t)+...
        sqrt(0.1*Fs)*randn(1,length(t));
[Pxx,f] = pwelch(x,128,[],4096,Fs,'onesided');
```



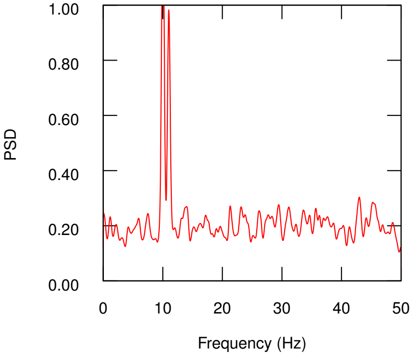
- Length of data sequence must be increased.
- Still can't resolve the two frequencies as the window size is too small.

pwelch Function - cos + WGN signal

```
Fs = 100;   t = 0:1/Fs:50;
x = cos(2*pi*10*t)+cos(2*pi*11*t)+...
        sqrt(0.1*Fs)*randn(1,length(t));
[Pxx,f] = pwelch(x,256,[],4096,Fs,'onesided');
```

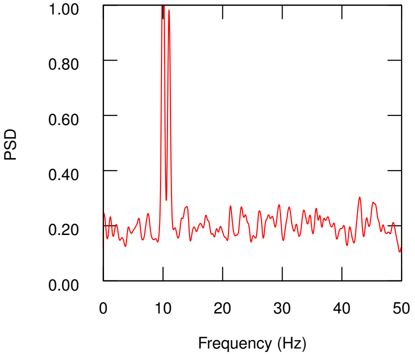
pwelch Function - cos + WGN signal

```
Fs = 100;    t = 0:1/Fs:50;
x = cos(2*pi*10*t)+cos(2*pi*11*t)+...
        sqrt(0.1*Fs)*randn(1,length(t));
[Pxx,f] = pwelch(x,256,[],4096,Fs,'onesided');
```



pwelch Function - cos + WGN signal

```
Fs = 100;   t = 0:1/Fs:50;
x = cos(2*pi*10*t)+cos(2*pi*11*t)+...
        sqrt(0.1*Fs)*randn(1,length(t));
[Pxx,f] = pwelch(x,256,[],4096,Fs,'onesided');
```



- Now we can resolve the two frequencies.

Spectral Estimation - Remarks

- The length of the data sequence determines the maximum resolution that can be observed.
- Increasing the window length of each segment in the data increases the resolution.
- Decreasing the window length of each segment in the data decreases the variance of the estimate.
- `nFFT` only affects the amount of details shown and not the resolution.