Power Spectral Density and Correlation*

In an analogy to the energy signals, let us define a function that would give us some indication of the relative power contributions at various frequencies, as $S_f(\omega)$. This function has units of power per Hz and its integral yields the power in $f(t)$ and is known as *power spectral density* function. Mathematically,

$$P = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_f(\omega) d\omega. \tag{1}$$

Assume that we are given a signal $f(t)$ and we truncate it over the interval $(-T/2, T/2)$. This truncated version is $f(t)\Pi(t/T)$. If $f(t)$ is finite over the interval $(-T/2, T/2)$, then the truncated function $f(t)\Pi(t/T)$ has finite energy and its Fourier transform $F_T(\omega)$ is

$$F_T(\omega) = \mathcal{F}\{f(t)\Pi(t/T)\}. \tag{2}$$

Parseval's theorem of the truncated version is

$$\int_{-T/2}^{T/2} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F_T(\omega)|^2 d\omega. \tag{3}$$

Therefore, the average power P across a one-ohm resistor is given by

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T} \frac{1}{2\pi} \int_{-\infty}^{\infty} |F_T(\omega)|^2 d\omega. \tag{4}$$

Combining Equations (1) and (4), we get

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_f(\omega) d\omega = \lim_{T\to\infty} \frac{1}{T} \frac{1}{2\pi} \int_{-\infty}^{\infty} |F_T(\omega)|^2 d\omega. \tag{5}$$

In addition if we insist that this relation should hold over each frequency increment, then

$$M_f(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\omega} S_f(u) du = \lim_{T\to\infty} \frac{1}{T} \frac{1}{2\pi} \int_{-\infty}^{\omega} |F_T(u)|^2 du. \tag{6}$$

$M_f(\omega)$ is known as the *cumulative power spectrum.* Now, interchange the order of the limiting operator and the integration (assuming it is valid)

$$2\pi M_f(\omega) = \int_{-\infty}^{\omega} S_f(u) du = \int_{-\infty}^{\omega} \lim_{T\to\infty} \frac{|F_T(u)|^2}{T} du. \tag{7}$$

If $M_f(\omega)$ is differentiable, then

$$2\pi \frac{dM_f(\omega)}{d\omega} = S_f(\omega). \tag{8}$$

Under these conditions

$$S_f(\omega) = \lim_{T\to\infty} \frac{|F_T(\omega)|^2}{T} \tag{9}$$

*This derivation are adapted from F.G. Stremler, *Introduction to Communication Systems*, 2nd Ed., Addison-Wesley, Massachusetts, 1982

Taking the inverse Fourier transform of Equation (9) gives us

$$\mathcal{F}^{-1}\{S_f(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{|F_T(\omega)|^2}{T} e^{j\omega\tau} d\omega. \tag{10}$$

Interchanging the order of operation yields

$$\begin{aligned} \mathcal{F}^{-1}\{S_f(\omega)\} &= \lim_{T\to\infty} \frac{1}{2\pi T}\int_{-\infty}^{\infty} F_T^*(\omega)F_T(\omega)e^{j\omega\tau}d\omega \\ &= \lim_{T\to\infty} \frac{1}{2\pi T}\int_{-\infty}^{\infty}\int_{-T/2}^{T/2} f^*(t)e^{j\omega t}dt\int_{-T/2}^{T/2} f(t)e^{-j\omega t'}dt' e^{j\omega\tau}d\omega \\ &= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} f^*(t)\int_{-T/2}^{T/2} f(t')\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(t-t'+\tau)}d\omega\right]dt'dt. \end{aligned} \tag{11}$$

The integration over ω in the above equation is equal to $\delta(t - t' + \tau)$, therefore

$$\begin{aligned} \mathcal{F}^{-1}\{S_f(\omega)\} &= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} f^*(t)\int_{-T/2}^{T/2} f(t')\delta(t-t'+\tau)dt'dt \\ &= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} f^*(t)f(t+\tau)dt \end{aligned} \tag{12}$$

The inverse Fourier transform of $S_f(\omega)$ is called *autocorrelation function* of $f(t)$ and is denoted by $R_f(\tau)$.

To summarize

$$R_f(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} f^*(t)f(t+\tau)dt \tag{13}$$

and

$$S_f(\omega) = \mathcal{F}\{R_f(\tau)\} \tag{14}$$

If the signal is periodic with period T_0 then,

$$R_f(\tau) = \frac{1}{T_0}\int_{-T_0} f^*(t)f(t+\tau)dt \tag{15}$$