EE 451

Solution of Difference Equations in the Time Domain

The difference equation

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]$$

has the solution

$$y[n] = \underbrace{\{\alpha_1 \lambda_1^n + \cdots + \alpha_N \lambda_N^n\}\,\mu[n] + \gamma_0 \delta[n] + \cdots + \gamma_{M-N-1}\delta[n-(M-N-1)]}_{y_{tr}[n]} + \underbrace{y_p[n]}_{y_{ss}[n]} \qquad (1)$$

where the λ_k's are the roots of the characteristic polynomial of the system:

$$d_0 \lambda^N + d_1 \lambda^{N-1} + \cdots + d_{N-1}\lambda + d_N = 0.$$

(For repeated roots, use λ_k^n, $n\lambda_k^n$, $n^2\lambda_k^n$, etc. For example, if $\lambda_1 = \lambda_2$, you would use $\alpha_1\lambda_1^n + \alpha_2 n\lambda_1^n$ instead of $\alpha_1\lambda_1^n + \alpha_2\lambda_1^n$.)

$y_p[n]$ is the particular (steady-state) solution which depends on the input:

$x[n]$	$y_p[n]$
$A\mu[n]$	$\beta\mu[n]$
$Aa^n\mu[n]$	$\beta a^n \mu[n]$
$A\cos[\omega_o n] + B\sin[\omega_o n]$	$\beta_1\cos[\omega_o n] + \beta_2\sin[\omega_o n]$

If the characteristic polynomial has a root at the value of an input exponential (e.g., $\lambda_1 = \frac{1}{2}$ and $x[n] = \left(\frac{1}{2}\right)^n$) you would use $\beta n\lambda_k^n$ for $y_p[n]$ (e.g., $y_p[n] = \beta n\left(\frac{1}{2}\right)^n$).

There are $M-N$ γ_i's. If $M \leq N$ there are no γ_i's.

To find the impulse response, there is no $y_p[n]$, and there are $M-N+1$ γ_i's. If $M < N$ there are no γ_i's for the impulse response.

Solve for the unknowns by finding $y[0]$, $y[1]$, $y[2]$, $\cdots$ until you get as many equations as you have unknowns. Solve these equations for the unknowns.

The *transient response* of the system is $y_{tr}[n] = \alpha_1\lambda_1^n + \cdots + \alpha_N\lambda_N^n + \gamma_1\delta[n] + \cdots + \gamma_{M-N}\delta[n-(M-N])$ – it is the part of the response which dies out in time. The *steady-state response* of the system is $y_{ss}[n] = y_p[n]$ – it stays around as long as the input drives it. The total response is the sum of the two: $y[n] = y_{tr}[n] + y_{ss}[n]$.

You can solve the difference equation for the case $x[n] = 0$, subject to the initial conditions of the system. This is the *natural response* or *zero-input response*, $y_{zi}[n]$, of the system. You can further solve for the case $y[n] = 0$ for $n < 0$ for the actual input $x[n]$. This is the *forced response* or *zero-state response*, $y_{zs}[n]$, of the system. The total response is the sum of the two: $y[n] = y_{zi}[n] + y_{zs}[n]$.

A system is *BIBO stable* if, for every bounded input, the output is bounded. If the input $x[n]$ is bounded, then the particular solution $y_p[n]$ will be bounded. Thus, the only possible unbounded terms in Eq. (1) are the λ_k^n's. These terms are bounded ($\lambda_k^n \to 0$ as $n \to \infty$) if $|\lambda_k| < 1$. Hence, the system is BIBO stable if $|\lambda_k| < 1$ for all the k's. (λ_k^n doesn't blow up if $|\lambda_k| = 1$. However, if $|\lambda_k| = 1$, the input $x[n] = \lambda_k^n$ will produce the output $y[n] = \alpha_1 \lambda_k^n + \beta n \lambda_k^n$, and the $\beta n \lambda_k^n$ term will blow up as $n \to \infty$.)