

Lecture

Error Mechanization (NAV)

EE 565: Position, Navigation and Timing

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1 Attitude

NAV Attitude Error

$$\dot{\hat{C}}_b^n = C_b^n (\Omega_{ib}^b - \Omega_{in}^b) = \frac{d}{dt} \left[(\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \hat{C}_b^n \right] =$$

$$\begin{aligned} (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \hat{C}_b^n \Omega_{nb}^b &= [\delta\dot{\vec{\psi}}_{nb}^n \times] \hat{C}_b^n + (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \dot{\hat{C}}_b^n = \\ &\approx (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \hat{C}_b^n (\hat{\Omega}_{nb}^b + \delta\Omega_{ib}^b - \delta\Omega_{in}^b) \\ &\approx (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \hat{C}_b^n \hat{\Omega}_{nb}^b + \hat{C}_b^n (\delta\Omega_{ib}^b - \delta\Omega_{in}^b) \end{aligned}$$

$$[\delta\dot{\vec{\psi}}_{nb}^n \times] = \hat{C}_b^n (\delta\Omega_{nb}^b - \delta\Omega_{in}^b) \hat{C}_n^b = [\hat{C}_b^n (\delta\vec{\omega}_{ib}^b - \delta\vec{\omega}_{in}^b) \times] \quad (1)$$

$$\delta\dot{\vec{\psi}}_{nb}^n = \hat{C}_b^n (\delta\omega_{ib}^b - \delta\omega_{in}^b) \quad (2)$$

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NAV Attitude Error (cont.) Computing $\delta\vec{\omega}_{in}^b$

Recall that

$$\vec{\omega}_{in}^b = C_n^b \vec{\omega}_{in}^n$$

Expressing the above equation in terms of estimates we get

$$\begin{aligned} \hat{\vec{\omega}}_{in}^b + \delta\vec{\omega}_{in}^b &= \hat{C}_n^b (\mathcal{I} - [\delta\vec{\psi}_{nb}^n \times]) (\hat{\vec{\omega}}_{in}^n + \delta\vec{\omega}_{in}^n) \\ \delta\vec{\omega}_{in}^b &\approx \hat{C}_n^b (\delta\vec{\omega}_{in}^n - [\delta\vec{\psi}_{nb}^n \times] \hat{\vec{\omega}}_{in}^n) \end{aligned}$$

Substituting this result in Equation 2

$$\delta\dot{\vec{\psi}}_{nb}^n = -\hat{\Omega}_{in}^n \delta\vec{\psi}_{nb}^n + \hat{C}_b^n \delta\vec{\omega}_{ib}^b - \delta\vec{\omega}_{in}^n \quad (3)$$

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NAV Attitude Error (cont.) Computing $\delta\vec{\omega}_{in}^n$

Using Taylor series and retaining the first order terms

$$\delta\vec{\omega}_{in}^n = \frac{\partial\vec{\omega}_{in}^n}{\partial\hat{r}_{eb}^n}\delta\vec{r}_{eb}^n + \frac{\partial\vec{\omega}_{in}^n}{\partial\hat{v}_{eb}^n}\delta\vec{v}_{eb}^n$$

where $\vec{r}_{eb}^n = [L_b, \lambda_b, h_b]^T$ and

$$\hat{\omega}_{in}^n = \hat{\omega}_{ie}^n + \hat{\omega}_{en}^n = \begin{pmatrix} \omega_{ie} \cos \hat{L}_b + \frac{\hat{v}_{eb,E}^n}{R_E(\hat{L}_b) + \hat{h}_b} \\ -\frac{\hat{v}_{eb,N}^n}{R_N(\hat{L}_b) + \hat{h}_b} \\ -\omega_{ie} \sin \hat{L}_b - \frac{\hat{v}_{eb,E}^n \tan \hat{L}_b}{R_E(\hat{L}_b) + \hat{h}_b} \end{pmatrix}$$

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NAV Attitude Error (cont.) Define terms for simplicity

$$-\hat{\Omega}_{in}^n = -[(\hat{\omega}_{ie}^n + \hat{\omega}_{en}^n) \times] = F_{\psi\psi} \quad (4)$$

$$-\frac{\partial\hat{\omega}_{in}^n}{\partial\hat{v}_{eb}^n} = \begin{pmatrix} 0 & -\frac{1}{R_E(\hat{L}_b) + \hat{h}_b} & 0 \\ \frac{1}{R_N(\hat{L}_b) + \hat{h}_b} & 0 & 0 \\ 0 & \frac{\tan \hat{L}_b}{R_E(\hat{L}_b) + \hat{h}_b} & 0 \end{pmatrix} = F_{\psi v} \quad (5)$$

$$-\frac{\partial\hat{\omega}_{in}^n}{\partial\hat{r}_{eb}^n} = \begin{pmatrix} \omega_{ie} \sin \hat{L}_b & 0 & \frac{\hat{v}_{eb,E}^n}{(R_E(\hat{L}_b) + \hat{h}_b)^2} \\ 0 & 0 & -\frac{\hat{v}_{eb,N}^n}{(R_N(\hat{L}_b) + \hat{h}_b)^2} \\ \omega_{ie} \cos \hat{L}_b + \frac{\hat{v}_{eb,E}^n}{(R_E(\hat{L}_b) + \hat{h}_b) \cos^2 \hat{L}_b} & 0 & -\frac{\hat{v}_{eb,E}^n \tan \hat{L}_b}{(R_E(\hat{L}_b) + \hat{h}_b)^2} \end{pmatrix} = F_{\psi r} \quad (6)$$

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NAV Attitude Error (cont.) Final Expression

$$\delta\vec{\psi}_{nb}^n = F_{\psi\psi}\delta\vec{\psi}_{nb}^n + F_{\psi v}\delta\vec{v}_{eb}^n + F_{\psi r}\delta\vec{r}_{eb}^n + \hat{C}_b^n \delta\vec{\omega}_{ib}^b \quad (7)$$

where $F_{\psi\psi}$, $F_{\psi v}$ and $F_{\psi r}$ are defined by Equations 4, 5 and 6, respectively.

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2 NAV Velocity Error

Velocity

$$\dot{\vec{v}}_{eb}^n = C_b^n \vec{f}_{ib}^b + \vec{g}_b^n - (\Omega_{en}^n + 2\Omega_{ie}^n) \vec{v}_{eb}^e \quad (8)$$

$$\begin{aligned} \dot{\vec{v}}_{eb}^n &= \hat{C}_b^n \hat{f}_{ib}^b + \hat{g}_b^n - (\hat{\Omega}_{en}^n + 2\hat{\Omega}_{ie}^n) \hat{\vec{v}}_{eb}^n \\ &= (\mathcal{I} - [\delta\vec{\psi}_{nb}^n \times]) C_b^n (\vec{f}_{ib}^b - \delta\vec{f}_{ib}^b) + \hat{g}_b^n - (\hat{\Omega}_{en}^n + 2\hat{\Omega}_{ie}^n) \hat{\vec{v}}_{eb}^n \end{aligned} \quad (9)$$

Computing the $\delta\vec{v}_{eb}^n$ we obtain

$$\begin{aligned} \delta\dot{\vec{v}}_{eb}^n &= \dot{\vec{v}}_{eb}^n - \hat{\vec{v}}_{eb}^n \\ &= [\delta\vec{\psi}_{nb}^n \times] C_b^n \vec{f}_{ib}^b + \hat{C}_b^n \delta\vec{f}_{ib}^b + \delta\vec{g}_b^n + \\ &\quad - (\hat{\Omega}_{en}^n + 2\hat{\Omega}_{ie}^n) \delta\vec{v}_{eb}^n - (\delta\Omega_{en}^n + 2\delta\Omega_{ie}^n) \hat{\vec{v}}_{eb}^n \end{aligned}$$

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Velocity Cont. $-(\hat{\Omega}_{en}^n + 2\hat{\Omega}_{ie}^n)$

The term $-(\hat{\Omega}_{en}^n + 2\hat{\Omega}_{ie}^n)$ is derived as

$$-[(\hat{\omega}_{en}^n + 2\hat{\omega}_{ie}^n) \times] = \begin{pmatrix} 0 & -2\omega_{ie} \sin \hat{L}_b - \frac{\hat{v}_{eb,E}^n \tan \hat{L}_b}{R_E(\hat{L}_b) + \hat{h}_b} & \frac{\hat{v}_{eb,N}^n}{R_N(\hat{L}_b) + \hat{h}_b} \\ 2\omega_{ie} \sin \hat{L}_b + \frac{\hat{v}_{eb,E}^n \tan \hat{L}_b}{R_E(\hat{L}_b) + \hat{h}_b} & 0 & 2\omega_{ie} \cos \hat{L}_b + \frac{\hat{v}_{eb,E}^n}{R_E(\hat{L}_b) + \hat{h}_b} \\ -\frac{\hat{v}_{eb,N}^n}{R_N(\hat{L}_b) + \hat{h}_b} & -2\omega_{ie} \cos \hat{L}_b - \frac{\hat{v}_{eb,E}^n}{R_E(\hat{L}_b) + \hat{h}_b} & 0 \end{pmatrix} = F_{\Omega\Omega} \quad (10)$$

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Velocity Cont. $-(\delta\hat{\Omega}_{en}^n + 2\delta\hat{\Omega}_{ie}^n)\hat{v}_{eb}^n$

The term $-(\delta\hat{\Omega}_{en}^n + 2\delta\hat{\Omega}_{ie}^n)\hat{v}_{eb}^n$ is derived as

$$-(\delta\hat{\Omega}_{en}^n + 2\delta\hat{\Omega}_{ie}^n)\hat{v}_{eb}^n = \frac{\partial F_{\Omega\Omega}}{\partial \hat{r}_{eb}^n} \delta \hat{r}_{eb}^n \hat{v}_{eb}^n + \frac{\partial F_{\Omega\Omega}}{\partial \hat{v}_{eb}^n} \delta \hat{v}_{eb}^n \hat{v}_{eb}^n \quad (11)$$

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Velocity Cont. Gravity Error

$$\delta \vec{g}_b^n \approx \frac{2g_0(\hat{L}_b)}{r_{eS}^e(\hat{L}_b)} \delta h_b \quad (12)$$

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Velocity Error Cont. After tons of algebra

$$\delta \dot{\vec{v}}_{eb}^n = F_{v\psi} \delta \vec{\psi}_{nb}^n + F_{vv} \delta \vec{v}_{eb}^n + F_{vr} \delta \vec{r}_{eb}^n + \hat{C}_b^n \delta \vec{f}_{ib}^b \quad (13)$$

where $F_{v\psi}$, F_{vv} and F_{vr} are defined by Equations 14, 15 and 16, respectively.

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Velocity Cont. $F_{v\psi}$, F_{vv} and F_{vr}

$$F_{v\psi} = -[(\hat{C}_b^n \hat{f}_{ib}^b) \times] \quad (14)$$

$$\mathbf{F}_{vv} = \begin{pmatrix} \frac{\hat{v}_{eb,D}^n}{(R_N(\hat{L}_b) + \hat{h}_b)} & -\frac{2\hat{v}_{eb,E}^n \tan \hat{L}_b}{(R_E(\hat{L}_b) + \hat{h}_b)} - 2\omega_{ie} \sin \hat{L}_b & \frac{\hat{v}_{eb,N}^n}{R_N(\hat{L}_b) + \hat{h}_b} \\ \frac{\hat{v}_{eb,E}^n \tan \hat{L}_b}{(R_E(\hat{L}_b) + \hat{h}_b)} + 2\omega_{ie} \sin \hat{L}_b & \frac{\hat{v}_{eb,N}^n \tan \hat{L}_b + \hat{v}_{eb,D}^n}{R_N(\hat{L}_b) + \hat{h}_b} & \frac{\hat{v}_{eb,E}^n}{(R_E(\hat{L}_b) + \hat{h}_b)} + 2\omega_{ie} \cos \hat{L}_b \\ -\frac{2\hat{v}_{eb,N}^n}{R_N(\hat{L}_b) + \hat{h}_b} & -\frac{2\hat{v}_{eb,E}^n}{(R_E(\hat{L}_b) + \hat{h}_b)} - 2\omega_{ie} \cos \hat{L}_b & 0 \end{pmatrix} \quad (15)$$

$$\mathbf{F}_{vr} = \begin{pmatrix} -\frac{(\hat{v}_{eb,E}^n)^2 \sec^2 \hat{L}_b}{R_E(\hat{L}_b) + \hat{h}_b} - 2\hat{v}_{eb,E}^n \omega_{ie} \cos \hat{L}_b & 0 & \frac{(\hat{v}_{eb,E}^n)^2 \tan \hat{L}_b}{(R_E(\hat{L}_b) + \hat{h}_b)^2} - \frac{\hat{v}_{eb,N}^n \hat{v}_{eb,D}^n}{(R_N(\hat{L}_b) + \hat{h}_b)^2} \\ \frac{\hat{v}_{eb,N}^n \hat{v}_{eb,E}^n \sec^2 \hat{L}_b}{R_E(\hat{L}_b) + \hat{h}_b} + 2\hat{v}_{eb,N}^n \omega_{ie} \cos \hat{L}_b - 2\hat{v}_{eb,D}^n \omega_{ie} \sin \hat{L}_b & 0 & -\frac{\hat{v}_{eb,N}^n \hat{v}_{eb,E}^n \tan \hat{L}_b + \hat{v}_{eb,E}^n \hat{v}_{eb,D}^n}{(R_E(\hat{L}_b) + \hat{h}_b)^2} \\ 2\hat{v}_{eb,E}^n \omega_{ie} \sin \hat{L}_b & 0 & \left(\frac{(\hat{v}_{eb,E}^n)^2}{(R_E(\hat{L}_b) + \hat{h}_b)^2} + \frac{(\hat{v}_{eb,N}^n)^2}{(R_N(\hat{L}_b) + \hat{h}_b)^2} - \frac{2g_0(\hat{L}_b)}{r_{eS}^e(\hat{L}_b)} \right) \end{pmatrix} \quad (16)$$

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3 Nav Position Error

Position

$$\dot{\vec{r}}_{eb}^n = \begin{pmatrix} \frac{\hat{v}_{eb,N}^n}{R_N(\hat{L}_b) + \hat{h}_b} \\ \frac{\hat{v}_{eb,E}^n}{\cos \hat{L}_b (R_E(\hat{L}_b) + \hat{h}_b)} \\ -\hat{v}_{eb,D}^n \end{pmatrix} \quad (17)$$

Computing $\delta \dot{\vec{r}}_{eb}^n$ using taylor series expansion and retaining only the first order terms

$$\delta \dot{\vec{r}}_{eb}^n = F_{r\psi} \delta \vec{\psi}_{nb}^n + F_{rv} \delta \vec{v}_{eb}^n + F_{rr} \delta \vec{r}_{eb}^n \quad (18)$$

where $F_{r\psi}$, F_{rv} and F_{rr} are defined by Equations 19, 20 and 21, respectively.

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Position Cont. $F_{r\psi}$, F_{rv} and F_{rr}

$$F_{r\psi} = 0_{3 \times 3} \quad (19)$$

$$\mathbf{F}_{rv} = \begin{pmatrix} \frac{1}{R_N(\hat{L}_b) + \hat{h}_b} & 0 & 0 \\ 0 & \frac{1}{(R_E(\hat{L}_b) + \hat{h}_b) \cos \hat{L}_b} & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (20)$$

$$\mathbf{F}_{rr} = \begin{pmatrix} 0 & 0 & -\frac{\hat{v}_{eb,N}^n}{(R_N(\hat{L}_b) + \hat{h}_b)^2} \\ \frac{\hat{v}_{eb,E}^n \tan \hat{L}_b}{(R_N(\hat{L}_b) + \hat{h}_b) \cos \hat{L}_b} & 0 & -\frac{\hat{v}_{eb,E}^n}{(R_N(\hat{L}_b) + \hat{h}_b)^2 \cos \hat{L}_b} \\ 0 & 0 & 0 \end{pmatrix} \quad (21)$$

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4 Summary

Summary - in terms of $\delta \vec{f}_{ib}^b, \delta \vec{\omega}_{ib}^b$

$$\begin{pmatrix} \delta \dot{\vec{\psi}}_{nb}^n \\ \delta \dot{\vec{v}}_{nb}^n \\ \delta \dot{\vec{r}}_{nb}^n \end{pmatrix} = \begin{pmatrix} F_{\psi\psi} & F_{\psi v} & F_{\psi r} \\ F_{v\psi} & F_{vv} & F_{vr} \\ 0_{3 \times 3} & F_{rv} & F_{rr} \end{pmatrix} \begin{pmatrix} \delta \vec{\psi}_{nb}^n \\ \delta \vec{v}_{nb}^n \\ \delta \vec{r}_{nb}^n \end{pmatrix} + \begin{pmatrix} 0 & \hat{C}_b^n \\ \hat{C}_b^n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \delta \vec{f}_{ib}^b \\ \delta \vec{\omega}_{ib}^b \end{pmatrix} \quad (22)$$

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A Basic Definitions

Notation Used

- *Truth value*

$$\vec{x}$$

- *Measured value*

$$\tilde{\vec{x}}$$

- *Estimated or computed value*

$$\hat{\vec{x}}$$

- *Error*

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

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B Linearization

Linearization using Taylor Series Expansion

Given a non-linear system $\dot{\vec{x}} = f(\vec{x}, t)$

Let's assume we have an estimate of $\vec{x}$, i.e., $\hat{\vec{x}}$ such that $\vec{x} = \hat{\vec{x}} + \delta\vec{x}$

$$\dot{\vec{x}} = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}} + \delta\vec{x}, t) \quad (23)$$

Using Taylor series expansion

$$\begin{aligned} f(\hat{\vec{x}} + \delta\vec{x}, t) &= \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}}, t) + \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} + H.O.T \\ &\approx \dot{\hat{\vec{x}}} + \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} \\ \Rightarrow \delta\dot{\vec{x}} &\approx \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} \end{aligned} \quad (24)$$

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C Inertial Measurements

Actual Measurements

Initially the accelerometer and gyroscope measurements, $\tilde{\vec{f}}_{ib}^b$ and $\tilde{\vec{\omega}}_{ib}^b$, respectively, will be modeled as

$$\tilde{\vec{f}}_{ib}^b = \vec{f}_{ib}^b + \Delta\vec{f}_{ib}^b \quad (25)$$

$$\tilde{\vec{\omega}}_{ib}^b = \vec{\omega}_{ib}^b + \Delta\vec{\omega}_{ib}^b \quad (26)$$

where $\vec{f}_{ib}^b$ and $\vec{\omega}_{ib}^b$ are the specific force and angular rates, respectively; and $\Delta\vec{f}_{ib}^b$ and $\Delta\vec{\omega}_{ib}^b$ represents the errors. In later lectures we will discuss more detailed description of these errors.

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Error Modeling Example

Accelerometers

$$\tilde{\vec{f}}_{ib}^b = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^b + n\vec{l}_a + \vec{w}_a \quad (27)$$

Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^b = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^b + G_g\vec{f}_{ib}^b + \vec{w}_g \quad (28)$$

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Pos, Vel, Force and Angular Rate Errors

- Position error

$$\delta\vec{r}_{\beta b}^\gamma = \vec{r}_{\beta b}^\gamma - \hat{\vec{r}}_{\beta b}^\gamma \quad (29)$$

- Velocity error

$$\delta\vec{v}_{\beta b}^\gamma = \vec{v}_{\beta b}^\gamma - \hat{\vec{v}}_{\beta b}^\gamma \quad (30)$$

- Specific force errors

$$\delta\vec{f}_{ib}^b = \vec{f}_{ib}^b - \hat{\vec{f}}_{ib}^b \quad (31)$$

$$\Delta_e\vec{f}_{ib}^b = \Delta\vec{f}_{ib}^b - \Delta\hat{\vec{f}}_{ib}^b = -\delta\vec{f}_{ib}^b \quad (32)$$

- Angular rate errors

$$\delta\vec{\omega}_{ib}^b = \vec{\omega}_{ib}^b - \hat{\vec{\omega}}_{ib}^b \quad (33)$$

$$\Delta_e\vec{\omega}_{ib}^b = \Delta\vec{\omega}_{ib}^b - \Delta\hat{\vec{\omega}}_{ib}^b = -\delta\vec{\omega}_{ib}^b \quad (34)$$

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D Attitude Error

Attitude Error Definition

Define

$$\delta C_b^\gamma = C_b^\gamma \hat{C}_\gamma^b = e^{[\delta\vec{\psi}_{\gamma b}^\gamma \times]} \approx \mathcal{I} + [\delta\vec{\psi}_{\gamma b}^\gamma \times] \quad (35)$$

This is the error in attitude resulting from errors in estimating the angular rates.

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Attitude Error Properties

The attitude error is a multiplicative small angle transformation from the actual frame to the computed frame

$$\hat{C}_b^\gamma = (\mathcal{I} - [\delta\vec{\psi}_{\gamma b}^\gamma \times])C_b^\gamma \quad (36)$$

Similarly,

$$C_b^\gamma = (\mathcal{I} + [\delta\vec{\psi}_{\gamma b}^\gamma \times])\hat{C}_b^\gamma \quad (37)$$

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E Estimates of Sensor Measurements

Specific Force and Angular Rates

Similarly we can attempt to estimate the specific force and angular rate by applying correction based on our estimate of the error.

$$\hat{\vec{f}}_{ib}^b = \tilde{\vec{f}}_{ib}^b - \Delta \hat{\vec{f}}_{ib}^b \quad (38)$$

$$\hat{\vec{\omega}}_{ib}^b = \tilde{\vec{\omega}}_{ib}^b - \Delta \hat{\vec{\omega}}_{ib}^b \quad (39)$$

where $\hat{\vec{f}}_{ib}^b$ and $\hat{\vec{\omega}}_{ib}^b$ are the accelerometer and gyroscope estimated calibration values, respectively.