EE 565: Position, Navigation and Timing
Navigation Mathematics: Kinematics (Coordinate Frame Transformation)

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Spring 2023

- Determine the detailed kinematic relationships between the 4 major frames of interest
  - The Earth–Centered Inertial (ECI) coordinate frame (i-frame)
  - The Earth–Centered Earth–Fixed (ECEF) coordinate frame (e-frame)
  - The Local Navigation (Nav) coordinate frame (n-frame)
  - The Body coordinate frame (b-frame)

- Relationship between the ECI and ECEF frames
  - ECI & ECEF have co-located orgins

$$\vec{r}_{ie} = \dot{\vec{r}}_{ie} = \ddot{\vec{r}}_{ie} = 0$$

  - The x, y, and z axis of the ECI & ECEF frames are coincident at time t_0
  - The ECEF frame rotates about the common z-axis at a fixed rate (ω_{ie})
    - Ignoring minor speed variatins (precession & nutation) $\omega_{ie} = 72.921151467\mu\text{rad/sec}$ (WGS84) which is $\approx 15°/\text{hr}$

- The angular velocity and acceleration are

$$\vec{\omega}\,^i_{ie} = \begin{bmatrix} 0 \\ 0 \\ \omega_{ie} \end{bmatrix} \qquad \dot{\vec{\omega}}\,^i_{ie} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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- The angle of rotation is

$$\theta_{ie} = \omega_{ie}(t - t_0)$$
$$= \omega_{ie}t + \theta_{GMST}$$

where GMST is the Greenwich mean sidereal time

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- The orientation of frame $\{e\}$ *wrt* frame $\{i\}$ becomes

$$C_e^i = R_{(\vec{z}, \theta_{ie})} = \begin{bmatrix} \cos\theta_{ie} & -\sin\theta_{ie} & 0 \\ \sin\theta_{ie} & \cos\theta_{ie} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Description of the navigation frame
  - Orientation of the *n*-frame *wrt* the *e*-frame

$$C_n^e = R_{(\vec{z}, \lambda_b)} R_{(\vec{y}, -L_b - 90°)}$$

$$= \begin{bmatrix} \cos \lambda_b & -\sin \lambda_b & 0 \\ \sin \lambda_b & \cos \lambda_b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin L_b & 0 & -\cos L_b \\ 0 & 1 & 0 \\ \cos L_b & 0 & -\sin L_b \end{bmatrix}$$

$$= \begin{bmatrix} -\sin L_b \cos \lambda_b & -\sin \lambda_b & -\cos L_b \cos \lambda_b \\ -\sin L_b \sin \lambda_b & \cos \lambda_b & -\cos L_b \sin \lambda_b \\ \cos L_b & 0 & -\sin L_b \end{bmatrix}$$

where geodetic Lat $= L_b$ and Geodetic Lon $= \lambda_b$
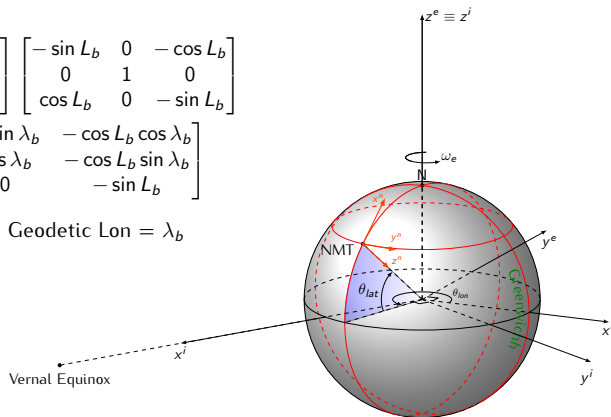
- Angular velocity of the *n*-frame *wrt* the *e*-frame resolved in the *e*-frame as a skew-symmetric matrix

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$$\Omega_{en}^e = \dot{C}_n^e \left[C_n^e\right]^T \qquad \boxed{\dot{C}_n^e = C_n^e \Omega_{en}^n = \Omega_{en}^e C_n^e}$$

$$= \begin{bmatrix} s_{L_b} s_{\lambda_b} \dot{\lambda}_b - c_{L_b} c_{\lambda_b} \dot{L}_b & -c_{\lambda_b} \dot{\lambda}_b & -c_{\lambda_b} s_{L_b} \dot{L}_b + c_{L_b} s_{\lambda_b} \dot{\lambda}_b \\ -c_{L_b} s_{\lambda_b} \dot{L}_b - c_{\lambda_b} s_{L_b} \dot{\lambda}_b & -s_{\lambda_b} \dot{\lambda}_b & s_{L_b} s_{\lambda_b} \dot{L}_b - c_{L_b} c_{\lambda_b} \dot{\lambda}_b \\ -s_{L_b} \dot{L}_b & 0 & -c_{L_b} \dot{L}_b \end{bmatrix} \left[C_n^e\right]^T$$

$$= \begin{bmatrix} 0 & -\dot{\lambda}_b & -\dot{L}_b \cos(\lambda_b) \\ \dot{\lambda}_b & 0 & -\dot{L}_b \sin(\lambda_b) \\ \dot{L}_b \cos(\lambda_b) & \dot{L}_b \sin(\lambda_b) & 0 \end{bmatrix}$$

- The angular velocity vector

$$\vec{\omega}\,_{en}^{e} = \begin{bmatrix} \sin(\lambda)\dot{L}_b \\ -\cos(\lambda)\dot{L}_b \\ \dot{\lambda}_b \end{bmatrix} \qquad \vec{\omega}\,_{en}^{n} = [C_n^e]^T\,\vec{\omega}\,_{en}^{e} = \begin{bmatrix} \cos(\lambda)\dot{\lambda}_b \\ -\dot{L}_b \\ -\sin(L_b)\dot{\lambda}_b \end{bmatrix}$$

- Hence the orientation of the *n*-frame *wrt* the *i*-frame becomes

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$$
C_n^i = C_e^i C_n^e = \begin{bmatrix} c_{\theta_{ie}} & -s_{\theta_{ie}} & 0 \\ s_{\theta_{ie}} & c_{\theta_{ie}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -s_{L_b}c_{\lambda_b} & -s_{\lambda_b} & -c_{L_b}c_{\lambda_b} \\ -s_{L_b}s_{\lambda_b} & c_{\lambda_b} & -c_{L_b}s_{\lambda_b} \\ c_{L_b} & 0 & -s_{L_b} \end{bmatrix}
$$

$$
= \begin{bmatrix} -\sin(L_b)\cos(\theta_{ie}+\lambda_b) & -\sin(\theta_{ie}+\lambda_b) & -\cos(L_b)\cos(\theta_{ie}+\lambda_b) \\ -\sin(L_b)\sin(\theta_{ie}+\lambda_b) & \cos(\theta_{ie}+\lambda_b) & -\cos(L_b)\sin(\theta_{ie}+\lambda_b) \\ \cos(L_b) & 0 & -\sin(L_b) \end{bmatrix}
$$

- The angular velocity of the *n*-frame *wrt* the *i*-frame resolved in the *i*-frame is

- The angular velocity of the *n*-frame *wrt* the *i*-frame resolved in the *i*-frame is

$$\vec{\omega}^{\,i}_{in} = \vec{\omega}^{\,i}_{ie} + C^i_e \vec{\omega}^{\,e}_{en}$$

$$= \begin{bmatrix} 0 \\ 0 \\ \omega_{ie} \end{bmatrix} + \begin{bmatrix} \cos(\theta_{ie}) & -\sin(\theta_{ie}) & 0 \\ \sin(\theta_{ie}) & \cos(\theta_{ie}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin(\lambda_b)\dot{L}_b \\ -\cos(\lambda_b)\dot{L}_b \\ \dot{\lambda}_b \end{bmatrix}$$

$$= \begin{bmatrix} \sin(\theta_{ie} + \lambda_b)\dot{L}_b \\ -\cos(\theta_{ie} + \lambda_b)\dot{L}_b \\ \omega_{ie} + \dot{\lambda}_b \end{bmatrix}$$

- The vector from the origin of the e-frame to the n-frame origin resolved in the e-frame (from the last lecture)

$$\vec{r}_{eb}^{\,e} = \begin{bmatrix} (R_E + h_b)\cos(L_b)\cos(\lambda_b) \\ (R_E + h_b)\cos(L_b)\sin(\lambda_b) \\ (R_E(1 - e^2) + h_b)\sin(L_b) \end{bmatrix} = \vec{r}_{en}^{\,e}$$

> Origins of the n-frame
> and the b-frame
> are the same

- The velocity of the n-frame *wrt* the e-frame resolved in the e-frame

$$\vec{v}_{en}^{\,e} = \frac{d}{dt}\vec{r}_{en}^{\,e} = \frac{\partial \vec{r}_{en}^{\,e}}{\partial L_b}\dot{L}_b + \frac{\partial \vec{r}_{en}^{\,e}}{\partial \lambda_b}\dot{\lambda}_b + \frac{\partial \vec{r}_{en}^{\,e}}{\partial h_b}\dot{h}_b$$

$$= \begin{bmatrix} -\sin(L_b)\cos(\lambda_b) & -\sin(\lambda_b) & -\cos(L_b)\cos(\lambda_b) \\ -\sin(L_b)\sin(\lambda_b) & \cos(\lambda_b) & -\cos(L_b)\sin(\lambda_b) \\ \cos(L_b) & 0 & -\sin(L_b) \end{bmatrix} \begin{bmatrix} (R_N + h_b)\dot{L}_b \\ \cos(L_b)(R_E + h_b)\dot{\lambda}_b \\ -\dot{h}_b \end{bmatrix}$$

- Recalling the form of C_n^e suggests that

$$\vec{v}_{en}^{\,e} = C_n^e \begin{bmatrix} (R_N + h_b)\dot{L}_b \\ \cos(L_b)(R_E + h_b)\dot{\lambda}_b \\ -\dot{h}_b \end{bmatrix} = C_n^e \vec{v}_{en}^{\,n}$$

- and hence,

$$\vec{v}_{en}^{\,n} = \begin{bmatrix} (R_N + h_b)\dot{L}_b \\ \cos(L_b)(R_E + h_b)\dot{\lambda}_b \\ -\dot{h}_b \end{bmatrix}$$

- Restating $\vec{v}_{en}^{\,n}$ as

$$\vec{v}_{en}^{\,n} = \begin{bmatrix} (R_N + h_b)\dot{L}_b \\ \cos(L_b)(R_E + h_b)\dot{\lambda}_b \\ -\dot{h}_b \end{bmatrix} = \begin{bmatrix} v_{en,N}^n \\ v_{en,E}^n \\ v_{en,D}^n \end{bmatrix}$$

- and recalling that

$$\vec{\omega}_{en}^{\,n} = \begin{bmatrix} \cos(L_b)\dot{\lambda}_b \\ -\dot{L}_b \\ -\sin(L_b)\dot{\lambda}_b \end{bmatrix}$$
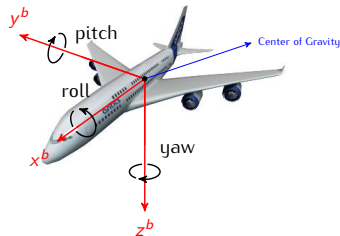
- suggests that

$$\vec{\omega}_{en}^{\,n} = \begin{bmatrix} \vec{v}_{en,E}^{\,n}/(R_E + h_b) \\ -\vec{v}_{en,N}^{\,n}/(R_N + h_b) \\ -\tan(L_b)\vec{v}_{en,E}^{\,n}/(R_E + h_b) \end{bmatrix}$$

- Description *wrt* the body frame
  - Orientation of the *b*-frame *wrt* the *n*-frame in terms of relative yaw (ψ), pitch (θ), then roll (ϕ) angles

$$C_b^n = R_{(\vec{z},\psi)} R_{(\vec{y},\theta)} R_{(\vec{x},\phi)} = \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi & c_\phi \end{bmatrix}$$

$$= \begin{bmatrix} c_\theta c_\psi & c_\psi s_\theta s_\phi - c_\phi s_\psi & c_\phi c_\psi s_\theta + s_\phi s_\psi \\ c_\theta s_\psi & c_\phi c_\psi + s_\theta s_\phi s_\psi & c_\phi s_\theta s_\psi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix}$$

- The angular velocity of the b–frame *wrt* the i–frame resolved/coordinatized in the i–frame

- The angular velocity of the *b*-frame *wrt* the *i*-frame resolved/coordinatized in the *i*-frame

$$\vec{\omega}_{ib}^{\,i} = \vec{\omega}_{in}^{\,i} + C_n^i \vec{\omega}_{nb}^{\,n}$$
$$= \vec{\omega}_{ie}^{\,i} + C_e^i \vec{\omega}_{en}^{\,e} + C_n^i \vec{\omega}_{nb}^{\,n}$$

- Position vectors to the origin of the body frame
  - The origins of the body and Nav frames are co-incident

  $$\vec{r}_{nb} = \vec{0}$$

  - The origins of the ECI and ECEF frames are co-incident

  $$\vec{r}_{eb} = \vec{r}_{ib} = \vec{r}_{en} = \vec{r}_{in}$$

  - Velocity of the b-frame wrt the i-frame resolved in the i-frame
    - A moving point in a rotation frame

  $$\vec{v}^{\,i}_{ib} = \frac{d}{dt}\vec{r}^{\,i}_{ib} = \frac{d}{dt}C^i_e\vec{r}^{\,e}_{eb}$$

  $$= C^i_e\Omega^e_{ie}\vec{r}^{\,e}_{eb} + C^i_e\vec{v}^{\,e}_{eb}$$

  $$= C^i_e\left(\Omega^e_{ie}\vec{r}^{\,e}_{eb} + \vec{v}^{\,e}_{eb}\right)$$

- Acceleration of the *b*-frame *wrt* the *i*-frame resolved in the *i*-frame
  - A moving point in a rotation frame

$$\vec{a}^{\,i}_{ib} = \frac{d}{dt}\vec{v}^{\,i}_{ib} = \frac{d}{dt}\left(C^i_e\left(\Omega^e_{ie}\vec{r}^{\,e}_{eb} + \vec{v}^{\,e}_{eb}\right)\right)$$

$$= \dot{C}^i_e\left(\Omega^e_{ie}\vec{r}^{\,e}_{eb} + \vec{v}^{\,e}_{eb}\right) + C^i_e\begin{pmatrix}\dot{\omega}=0\\ \cancel{\dot{\Omega}^e_{ie}}\,\vec{r}^{\,e}_{eb} + \Omega^e_{ie}\dot{\vec{r}}^{\,e}_{eb} + \dot{\vec{v}}^{\,e}_{eb}\end{pmatrix}$$

$$= C^i_e\Omega^e_{ie}\left(\Omega^e_{ie}\vec{r}^{\,e}_{eb} + \vec{v}^{\,e}_{eb}\right) + C^i_e\left(\Omega^e_{ie}\vec{v}^{\,e}_{eb} + \vec{a}^{\,e}_{eb}\right)$$

$$= C^i_e\left(\Omega^e_{ie}\Omega^e_{ie}\vec{r}^{\,e}_{eb} + 2\Omega^e_{ie}\vec{v}^{\,e}_{eb} + \vec{a}^{\,e}_{eb}\right)$$