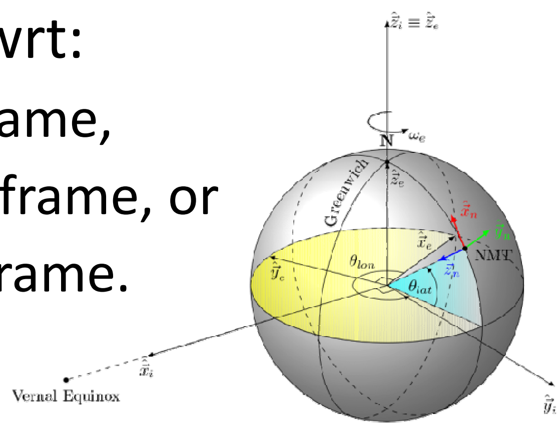
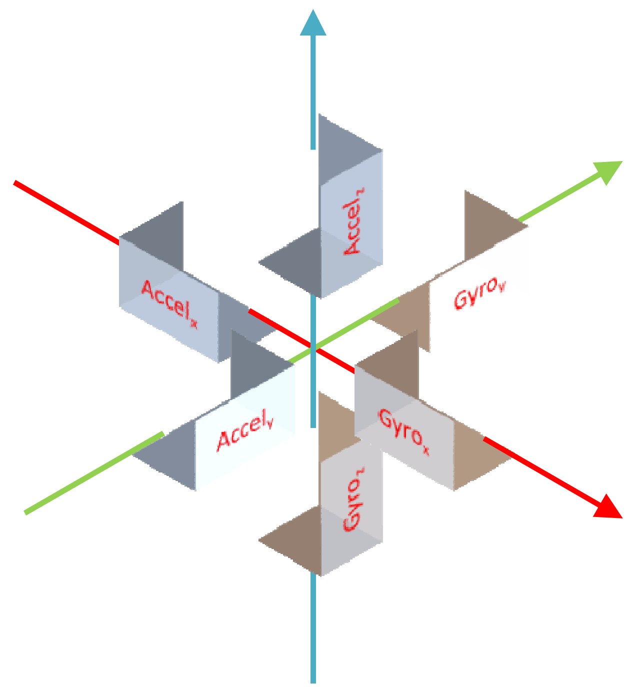
Inertial Navigation

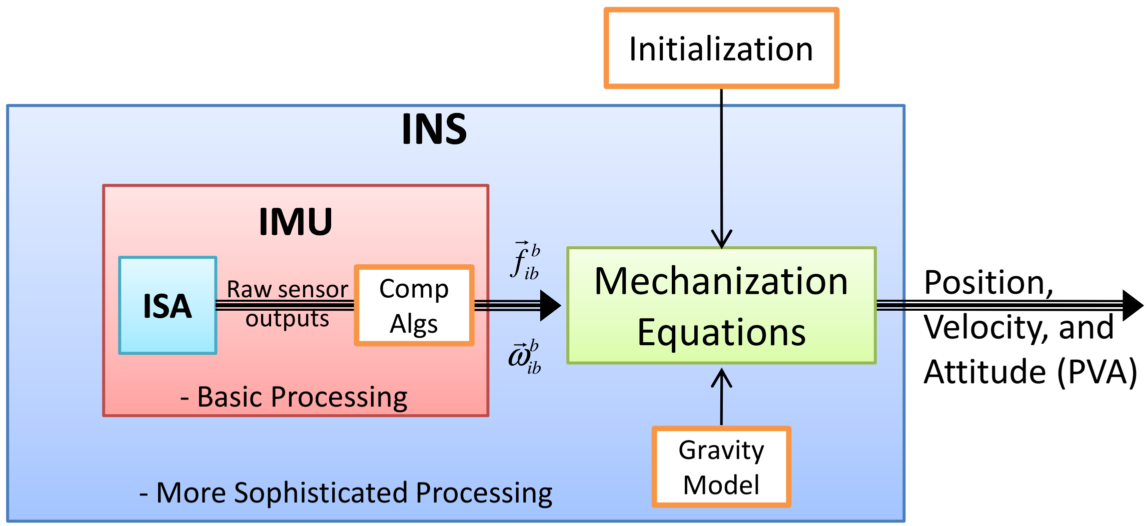
- Inertial Navigation
  - The process of "integrating" angular velocity & acceleration to determine One's position, velocity, and attitude (PVA)
    - Effectively "dead reckoning"
  - To measure the acceleration and angular velocity vectors we need at least 3-gyros and 3-accels
    - Typically configured in an orthogonal triad
  - The "mechanization" can be performed wrt:
    - The ECI frame,
    - The ECEF frame, or
    - The Nav frame.





Inertial Navigation

- An Inertial Navigation System (INS)
  - ISA – Inertial Sensor Assembly
    - Typically, 3-gyros + 3-accels + basic electronics (power, …)
  - IMU – Inertial Measurement Unit
    - ISA + Compensation algorithms (i.e. basic processing)
  - INS – Inertial Navigation System
    - IMU + gravity model + “mechanization” algorithms

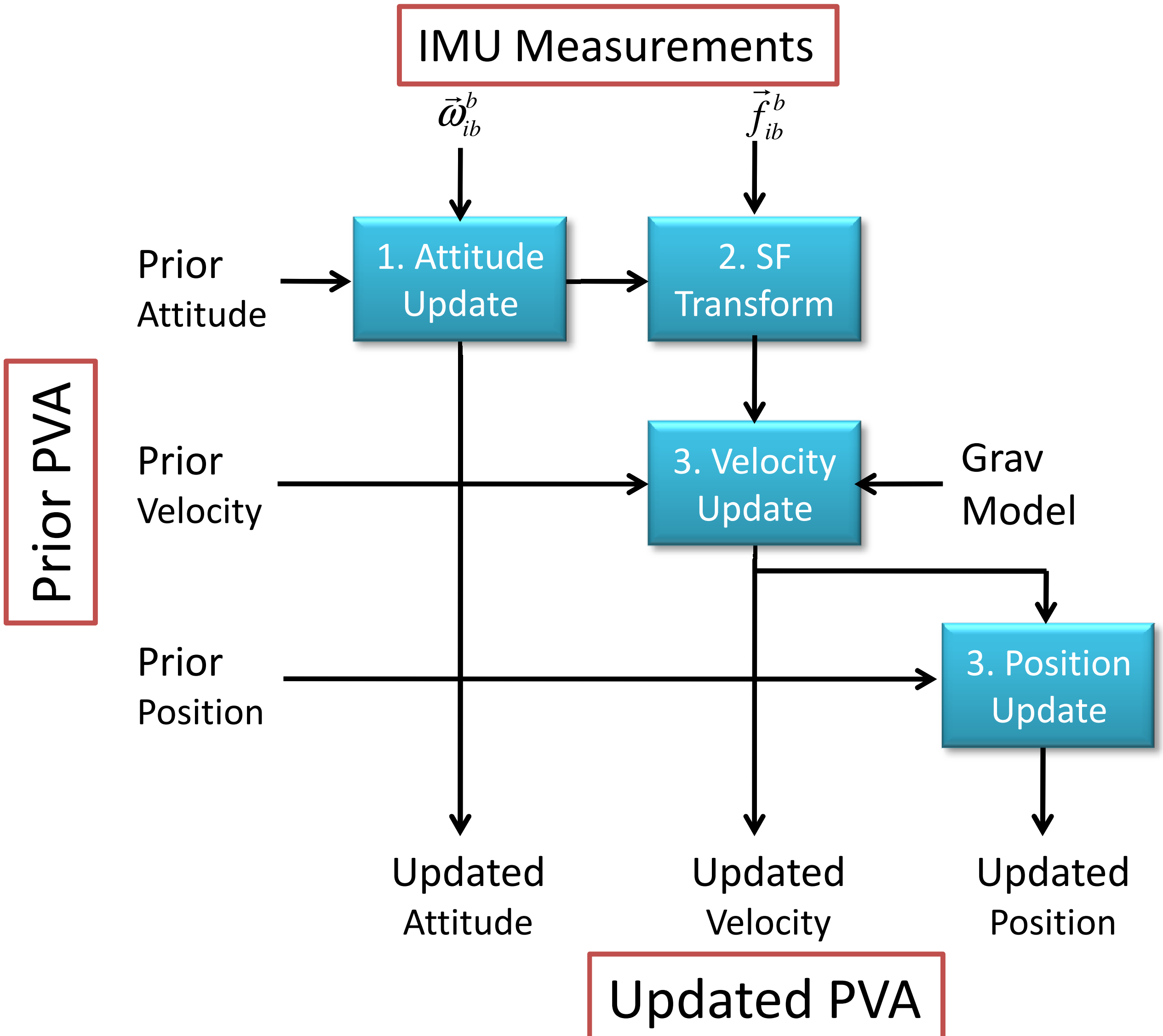


Inertial Navigation

A Four Step Mechanization

- Can be generically performed in four steps:
  1. Attitude Update
     - Update the prior attitude (a DCM, say) using the current angular velocity measurement
  2. Transform the specific force measurement
     - Typically, using the attitude computed in step 1.
  3. Update the velocity
     - Essentially integrate the result from step 2. with the use of a gravity model
  4. Update the Position
     - Essentially integrate the result from step 3.

Inertial Navigation
A Four Step Mechanization



Inertial Navigation

Case 1: ECI Frame Mechanization

- Determine our PVA wrt the ECI frame
  - Namely: Position $\vec{r}_{ib}^{\,i}$, Velocity, $\vec{v}_{ib}^{\,i}$, and attitude C_b^i

1. Attitude Update:

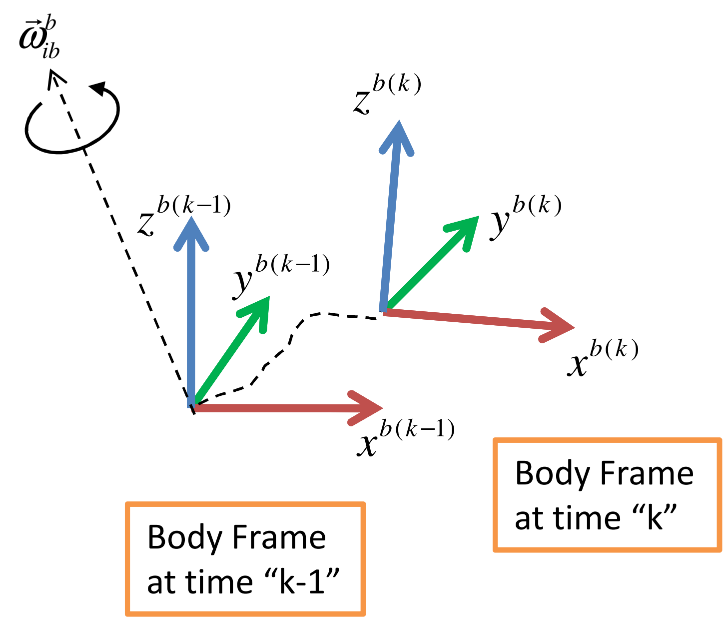
- Body orientation frame at time "k" wrt time "k-1"
  - Δt = Time k – Time k-1

$$C_{b(k)}^{i} = C_{b(k-1)}^{i} C_{b(k)}^{b(k-1)}$$

$$C_{b(k)}^{b(k-1)} = e^{\Omega_{ib}^{b}\Delta t}$$

$$C_b^i(+) = C_b^i(-) e^{\Omega_{ib}^{b}\Delta t}$$

$$\simeq C_b^i(-)\left(I + \left[\vec{\omega}_{ib}^{b} \times\right]\Delta t\right)$$



Inertial Navigation

Case 1: ECI Frame Mechanization

2. Specific Force Transformation

- Simply coordinatize the specific force

$$\vec{f}_{ib}^{\,i} = C_b^i(+)\ \vec{f}_{ib}^{\,b}$$

3. Velocity Update

- Assuming that we are in space (*i.e*. no centrifugal component)

$$\vec{\mathrm{a}}_{ib}^{\,i} = \vec{f}_{ib}^{\,i} - \vec{\gamma}_{ib}^{\,i}$$

- Thus, by simple numerical integration

$$\vec{v}_{ib}^{\,i}(+) = \vec{v}_{ib}^{\,i}(-) + \vec{\mathrm{a}}_{ib}^{\,i}\ \Delta t$$

4. Position Update

- By simple numerical integration

$$\vec{r}_{ib}^{\,i}(+) = \vec{r}_{ib}^{\,i}(-) + \vec{v}_{ib}^{\,i}(-)\Delta t + \vec{\mathrm{a}}_{ib}^{\,i}\ \frac{\Delta t^2}{2}$$

Inertial Navigation

Case 1: ECI Frame Mechanization

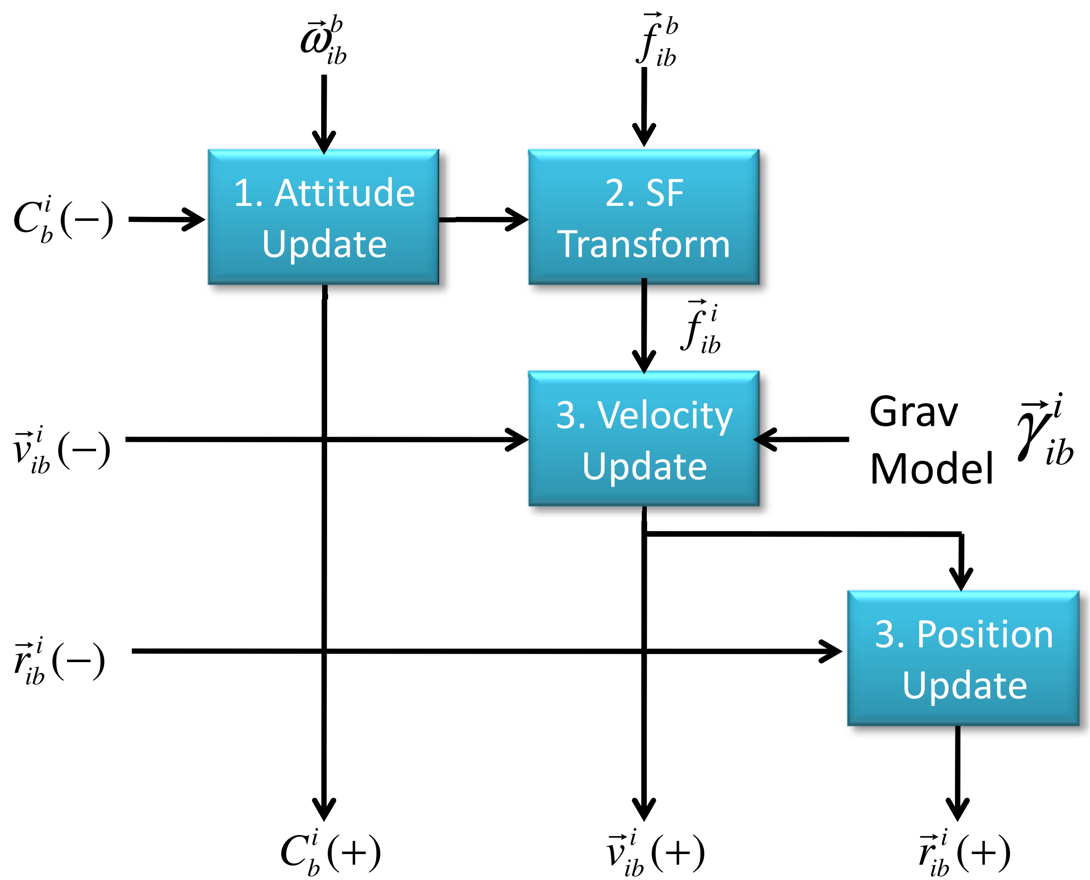
$$C_b^i(+) = C_b^i(-)\left(I + \left[\vec{\omega}_{ib}^b \times\right]\Delta t\right)$$

$$\vec{f}_{ib}^{\,i} = C_b^i(+)\ \vec{f}_{ib}^{\,b}$$

$$\vec{a}_{ib}^{\,i} = \vec{f}_{ib}^{\,i} - \vec{\gamma}_{ib}^{\,i}$$

$$\vec{v}_{ib}^{\,i}(+) = \vec{v}_{ib}^{\,i}(-) + \vec{a}_{ib}^{\,i}\ \Delta t$$

$$\vec{r}_{ib}^{\,i}(+) = \vec{r}_{ib}^{\,i}(-) + \vec{v}_{ib}^{\,i}(-)\Delta t + \vec{a}_{ib}^{\,i}\frac{\Delta t^2}{2}$$



Inertial Navigation

Case 2: ECEF Frame Mechanization

- Determine our PVA wrt the ECI frame
  - Namely: Position $\vec{r}_{eb}^{e}$, Velocity, $\vec{v}_{eb}^{e}$, and attitude C_b^e

1. Attitude Update:

- Start with the angular velocity $\vec{\omega}_{ib}^{?} = \vec{\omega}_{ie}^{?} + \vec{\omega}_{eb}^{?}$

$$\vec{\omega}_{ib}^{e} = \vec{\omega}_{ie}^{e} + \vec{\omega}_{eb}^{e}$$

$$\vec{\omega}_{eb}^{e} = C_b^e \vec{\omega}_{ib}^{b} - \vec{\omega}_{ie}^{i}$$

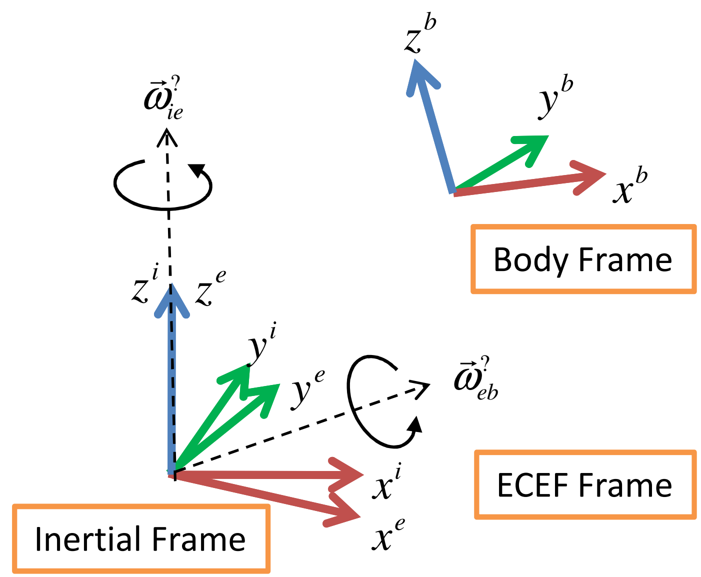
$$\left[(C\omega)\times\right] = C\left[\omega\times\right]C^T$$

$$\Omega_{eb}^{e} = C_b^e \Omega_{ib}^{b} C_e^b - \Omega_{ie}^{i}$$

$$C_b^e(+) = e^{\Omega_{eb}^{e}\Delta t} C_b^e(-)$$

$$\simeq \left(I + C_b^e \Omega_{ib}^{b} C_e^b \Delta t - \Omega_{ie}^{i}\Delta t\right) C_b^e(-)$$

$$= C_b^e(-)\left[I + \Omega_{ib}^{b}\Delta t\right] - \Omega_{ie}^{i} C_b^e(-)\Delta t$$



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Case 2: ECEF Frame Mechanization

2. Specific Force Transformation:

- Simply coordinatize the specific force

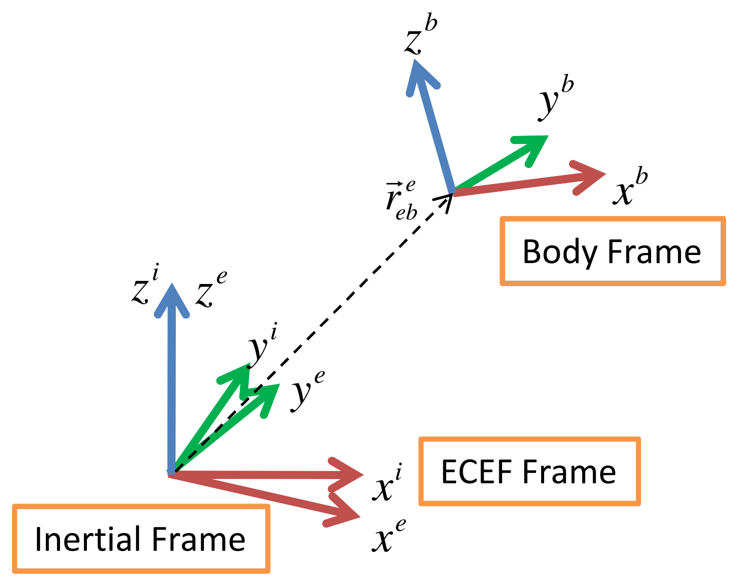
$$\vec{f}_{ib}^{\,e} = C_b^e(+)\;\vec{f}_{ib}^{\,b}$$

3. Velocity Update

- ECEF & ECI have the same origin

$$\vec{r}_{ib}^{\,i} = \cancelto{0}{\vec{r}_{ie}^{\,i}} + C_e^i \vec{r}_{eb}^{\,e} \quad \Rightarrow \vec{r}_{eb}^{\,e} = C_i^e \vec{r}_{ib}^{\,i}$$

$$\begin{aligned} \vec{v}_{eb}^{\,e} &= \dot{\vec{r}}_{eb}^{\,e} \\ &= \dot{C}_i^e \vec{r}_{ib}^{\,i} + C_i^e \dot{\vec{r}}_{ib}^{\,i} \\ &= \Omega_{ei}^e C_i^e \vec{r}_{ib}^{\,i} + C_i^e \vec{v}_{ib}^{\,i} \;= -\Omega_{ie}^e \vec{r}_{eb}^{\,e} + C_i^e \vec{v}_{ib}^{\,i} \end{aligned}$$



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Case 2: ECEF Frame Mechanization

$$\vec{a}^{e}_{eb} = \dot{\vec{v}}^{e}_{eb} = \frac{d}{dt}\left(-\Omega^{e}_{ie}\vec{r}^{e}_{eb} + C^{e}_{i}\vec{v}^{i}_{ib}\right)$$

$$= -\Omega^{e}_{ie}\dot{\vec{r}}^{e}_{eb} + \dot{C}^{e}_{i}\dot{\vec{r}}^{i}_{ib} + C^{e}_{i}\dot{\vec{v}}^{i}_{ib}$$

$$= -\Omega^{e}_{ie}\vec{v}^{e}_{eb} + \Omega^{e}_{ei}C^{e}_{i}\vec{v}^{i}_{ib} + C^{e}_{i}\vec{a}^{i}_{ib}$$

$$= -\Omega^{e}_{ie}\vec{v}^{e}_{eb} - \Omega^{e}_{ie}\left[\vec{v}^{e}_{eb} + \Omega^{e}_{ie}\vec{r}^{e}_{eb}\right] + C^{e}_{i}\vec{a}^{i}_{ib}$$

$$= -2\Omega^{e}_{ie}\vec{v}^{e}_{eb} - \Omega^{e}_{ie}\Omega^{e}_{ie}\vec{r}^{e}_{eb} + \vec{a}^{e}_{ib}$$

$$= -2\Omega^{i}_{ie}\vec{v}^{e}_{eb} + \vec{f}^{e}_{ib} + g^{e}_{b}$$

$$C^{e}_{i}\vec{v}^{i}_{ib} = \Omega^{e}_{ie}\vec{r}^{e}_{eb} + \vec{v}^{e}_{eb}$$

$$\vec{f}^{?}_{ib} = \vec{a}^{?}_{ib} - \gamma^{?}_{ib}$$

$$\vec{a}^{e}_{ib} = \vec{f}^{e}_{ib} + \gamma^{e}_{ib}$$

$$\vec{g}^{e}_{b} = \gamma^{e}_{ib} - \Omega^{e}_{ie}\Omega^{e}_{ie}\vec{r}^{e}_{eb}$$

$$\vec{a}^{e}_{ib} = \vec{f}^{e}_{ib} + g^{e}_{b} + \Omega^{e}_{ie}\Omega^{e}_{ie}\vec{r}^{e}_{eb}$$

$$\vec{v}^{e}_{eb}(+) = \vec{v}^{e}_{eb}(-) + \vec{a}^{e}_{eb}\Delta t$$

$$= \vec{v}^{e}_{eb}(-) + \left[\vec{f}^{e}_{ib} + \vec{g}^{e}_{b} - 2\Omega^{i}_{ie}\vec{v}^{e}_{eb}(-)\right]\Delta t$$

Inertial Navigation

Case 2: ECEF Frame Mechanization

4. Position Update

- By simple numerical integration

$$\vec{r}_{eb}^{e}(+) = \vec{r}_{eb}^{e}(-) + \vec{v}_{eb}^{e}(-)\Delta t + \vec{a}_{eb}^{e}\frac{\Delta t^{2}}{2}$$

Inertial Navigation

Case 2: ECEF Frame Mechanization

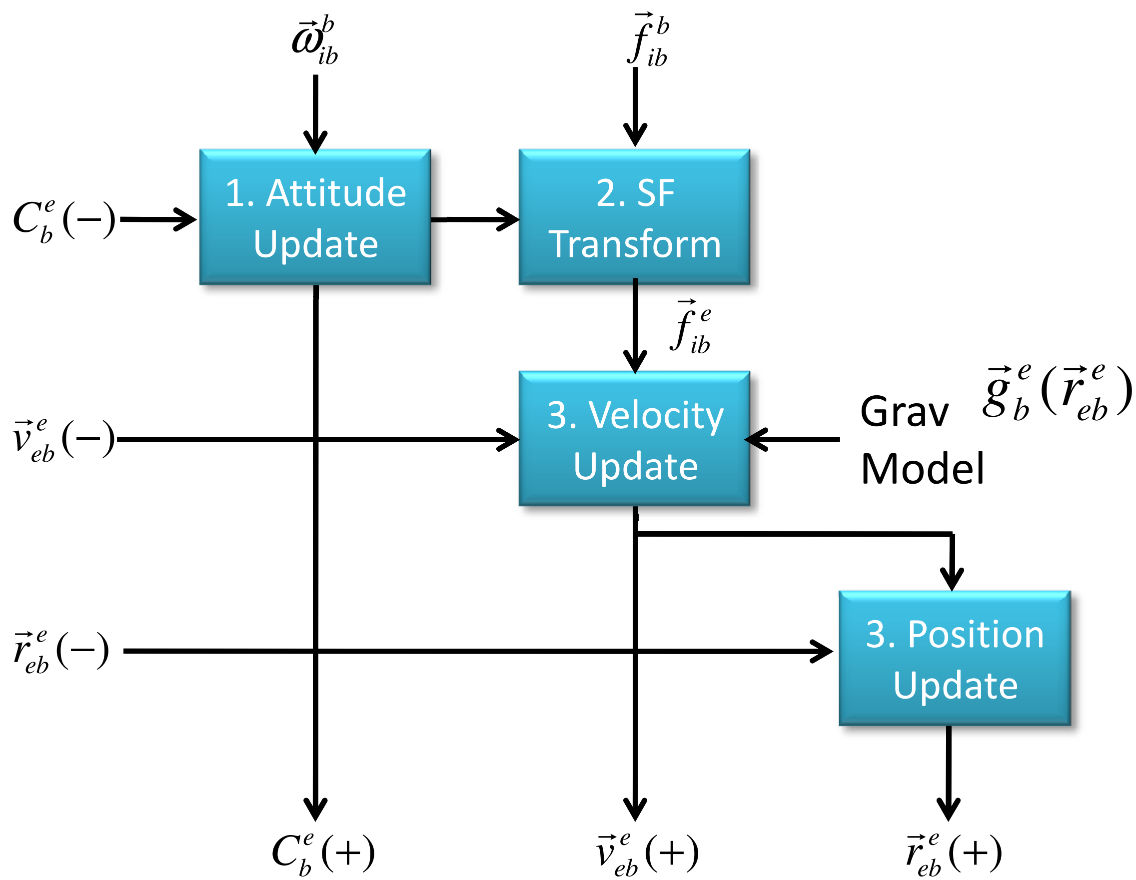
$$C_b^e(+) = C_b^e(-)\left[I + \Omega_{ib}^b \Delta t\right] - \Omega_{ie}^i C_b^e(-)\Delta t$$

$$\vec{f}_{ib}^{\,e} = C_b^e(+)\ \vec{f}_{ib}^{\,b}$$

$$\vec{a}_{eb}^{\,e} = \vec{f}_{ib}^{\,e} + \vec{g}_b^{\,e} - 2\Omega_{ie}^i \vec{v}_{eb}^{\,e}(-)$$

$$\vec{v}_{eb}^{\,e}(+) = \vec{v}_{eb}^{\,e}(-) + \vec{a}_{eb}^{\,e}\Delta t$$

$$\vec{r}_{eb}^{\,e}(+) = \vec{r}_{eb}^{\,e}(-) + \vec{v}_{eb}^{\,e}(-)\Delta t + \vec{a}_{eb}^{\,e}\frac{\Delta t^2}{2}$$



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Case 3: Navigation Frame Mechanization

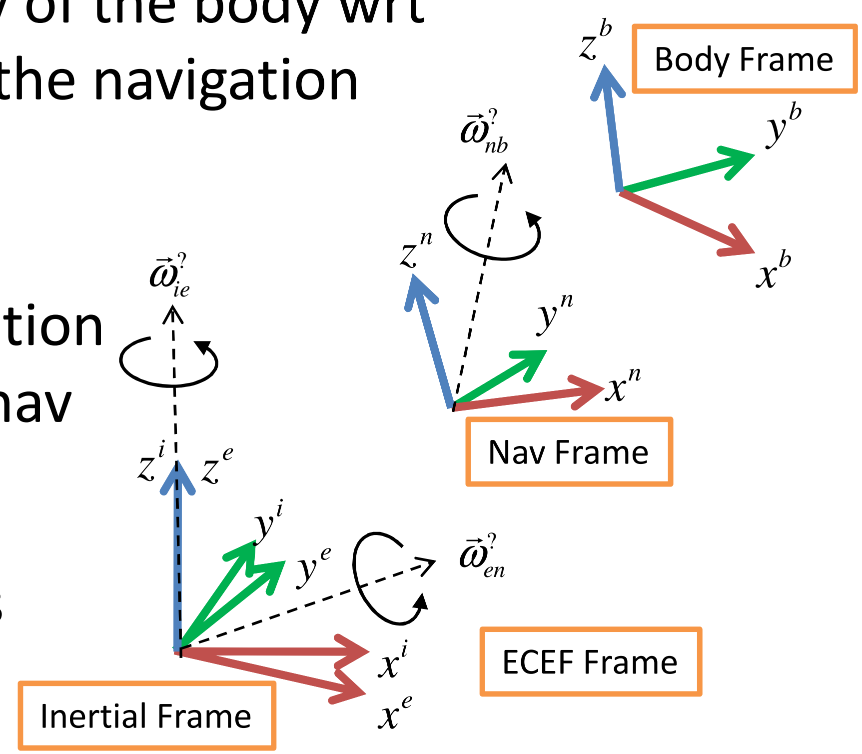
- Determine our PVA wrt the Nav frame
  - Position: Typically described in curvilinear coordinates

$$[L_b, \lambda_b, h_b]^T$$

  - Velocity: Typically the velocity of the body wrt the earth frame described in the navigation frame coords

$$\vec{v}_{eb}^{n}$$

  - Attitude: Typically the orientation of the body described in the nav frame C_b^n
  - This unusual choice facilitates the guidance system function



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Case 3: Navigation Frame Mechanization

1. Attitude Update:

- Note that $\vec{\omega}_{ib}^{?} = \vec{\omega}_{ie}^{?} + \vec{\omega}_{en}^{?} + \vec{\omega}_{nb}^{?} \quad \Rightarrow \vec{\omega}_{nb}^{b} = \vec{\omega}_{ib}^{b} - \vec{\omega}_{ie}^{b} - \vec{\omega}_{en}^{b}$
- Now

$$\dot{C}_b^n = C_b^n \, \Omega_{nb}^b = C_b^n \left(\Omega_{ib}^b - \Omega_{ie}^b - \Omega_{en}^b \right)$$

$$= C_b^n \, \Omega_{ib}^b - C_b^n \, \Omega_{ie}^b - C_b^n \, \Omega_{en}^b$$

$$= C_b^n \, \boxed{\Omega_{ib}^b} - \left(\boxed{\Omega_{ie}^n} + \boxed{\Omega_{en}^n} \right) C_b^n$$

See next slide

Measured by the gyro

$$\vec{\omega}_{ie}^n = C_e^n \, \vec{\omega}_{ie}^e$$

$$= \begin{bmatrix} * & * & * \\ * & * & * \\ \cos(L_b) & 0 & -\sin(L_b) \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ \omega_{ie} \end{bmatrix} = \omega_{ie} \begin{bmatrix} \cos(L_b) \\ 0 \\ -\sin(L_b) \end{bmatrix}$$

Inertial Navigation

Case 3: Navigation Frame Mechanization

- Last term: $\Omega^n_{en} \Leftrightarrow \vec{\omega}^n_{en}$

Courtesy of Mathematica

$$\dot{C}^e_n = C^e_n\,\Omega^n_{en} \Rightarrow \Omega^n_{en} = \left(C^e_n\right)^T \dot{C}^e_n = \begin{pmatrix} 0 & \sin(L_b)\dot{\lambda}_b & \boxed{\omega^n_{en,y}} \\ -\mathrm{si}\,\boxed{\omega^n_{en,z}}\,\dot{\lambda}_b & 0 & -\cos(L_b)\dot{\lambda}_b \\ \dot{L}_b & \mathrm{co}\,\boxed{\omega^n_{en,x}}\,\dot{\lambda}_b & 0 \end{pmatrix}$$

$$\vec{\omega}^n_{en} = \begin{bmatrix} \cos(L_b)\dot{\lambda}_b \\ -\dot{L}_b \\ -\sin(L_b)\dot{\lambda}_b \end{bmatrix}$$

From Eqn. 16 (Wedeward handout) $\Rightarrow$

$$\begin{pmatrix} \dot{L}_b \\ \dot{\lambda}_b \\ \dot{h}_b \end{pmatrix} = \begin{pmatrix} \dfrac{\vec{v}^n_{eb,N}}{\left(R_N + h_b\right)} \\ \dfrac{\vec{v}^n_{eb,E}}{\mathrm{Cos}(L_b)\left(R_E + h_b\right)} \\ -\vec{v}^n_{eb,D} \end{pmatrix}$$

$$\therefore \vec{\omega}^n_{en} = \begin{bmatrix} \dfrac{\vec{v}^n_{eb,E}}{\left(R_E + h_b\right)} \\ -\dfrac{\vec{v}^n_{eb,N}}{\left(R_N + h_b\right)} \\ -\dfrac{\tan(L_b)\vec{v}^n_{eb,E}}{\left(R_E + h_b\right)} \end{bmatrix}$$

$$C^n_b(+) \simeq C^n_b(-) + \Delta t \dot{C}^n_b$$
$$= C^n_b(-)\left(I + \Delta t \Omega^b_{ib}\right) - \left(\Omega^n_{ie} + \Omega^n_{en}\right) C^n_b(-)\Delta t$$

Inertial Navigation

Case 3: Nav Frame Mechanization

2. Specific Force Transformation:

- Simply coordinatize the specific force

$$\vec{f}_{ib}^{\,n} = C_b^n(+)\ \vec{f}_{ib}^{\,b}$$

3. Velocity Update

- Recall that $\left[(C\omega)\times\right] = C\left[\omega\times\right]C^T \Rightarrow C\left[\omega\times\right] = \left[(C\omega)\times\right]C$

$$\vec{v}_{eb}^{\,n} = C_e^n\ \vec{v}_{eb}^{\,e}$$

$$\begin{aligned}
\dot{\vec{v}}_{eb}^{\,n} &= \dot{C}_e^n\ \vec{v}_{eb}^{\,e} + C_e^n\ \dot{\vec{v}}_{eb}^{\,e} \\
&= \Omega_{ne}^n C_e^n\ \vec{v}_{eb}^{\,e} + C_e^n\left(\vec{f}_{ib}^{\,e} + \vec{g}_b^{\,e} - 2\Omega_{ie}^e \vec{v}_{eb}^{\,e}\right) \\
&= \vec{f}_{ib}^{\,n} + \vec{g}_b^{\,n} - \Omega_{en}^n\ \vec{v}_{eb}^{\,n} - 2C_e^n \Omega_{ie}^e \vec{v}_{eb}^{\,e} \\
&= \vec{f}_{ib}^{\,n} + \vec{g}_b^{\,n} - \Omega_{en}^n\ \vec{v}_{eb}^{\,n} - 2\Omega_{ie}^n C_e^n \vec{v}_{eb}^{\,e} \\
&= \vec{f}_{ib}^{\,n} + \vec{g}_b^{\,n} - \left(\Omega_{en}^n + 2\Omega_{ie}^n\right)\vec{v}_{eb}^{\,n}
\end{aligned}$$

Inertial Navigation

Case 3: Nav Frame Mechanization

- Finally

$$\vec{v}_{eb}^{\,n}(+) = \vec{v}_{eb}^{\,n}(-) + \Delta t\left[\vec{f}_{ib}^{\,n} + \vec{g}_{b}^{\,n} - \left(\Omega_{en}^{n} + 2\Omega_{ie}^{n}\right)\vec{v}_{eb}^{\,n}(-)\right]$$

4. Position Update

- Recalling the relationship between $\vec{v}_{eb}^{\,n}$ and the curvilinear coordinates

$$\begin{pmatrix}\dot{L}_b\\ \dot{\lambda}_b\\ \dot{h}_b\end{pmatrix} = \begin{pmatrix}\dfrac{\vec{v}_{eb,N}^{\,n}}{(R_N + h_b)}\\ \dfrac{\vec{v}_{eb,E}^{\,n}}{\mathrm{Cos}(L_b)(R_E + h_b)}\\ -\vec{v}_{eb,D}^{\,n}\end{pmatrix}$$

$$h_b(+) = h_b(-) + \Delta t\left[\vec{v}_{eb,D}^{\,n}(+)\right]$$

$$L_b(+) = L_b(-) + \Delta t\left[\frac{\vec{v}_{eb,N}^{\,n}(+)}{R_N - h_b}\right]$$

$$\lambda_b(+) = \lambda_b(-) + \Delta t\left[\frac{\vec{v}_{eb,E}^{\,n}(+)}{(R_E - h_b)\cos(L_b)}\right]$$

Inertial Navigation

Case 3: Nav Frame Mechanization

$$C_b^n(+) = C_b^n(-)\left(I + \Delta t \Omega_{ib}^b\right) - \left(\Omega_{ie}^n + \Omega_{en}^n\right) C_b^n(-)\Delta t$$

$$\vec{f}_{ib}^{\,n} = C_b^n(+)\, \vec{f}_{ib}^{\,b}$$

$$\vec{v}_{eb}^{\,n}(+) = \vec{v}_{eb}^{\,n}(-) + \Delta t\left[\vec{f}_{ib}^{\,n} + \vec{g}_b^{\,n} - \left(\Omega_{en}^n + 2\Omega_{ie}^n\right)\vec{v}_{eb}^{\,n}(-)\right]$$

$$h_b(+) = h_b(-) + \Delta t\left[\vec{v}_{eb,D}^{\,n}(+)\right]$$

$$L_b(+) = L_b(-) + \Delta t\left[\frac{\vec{v}_{eb,N}^{\,n}(+)}{R_N - h_b}\right]$$

$$\lambda_b(+) = \lambda_b(-) + \Delta t\left[\frac{\vec{v}_{eb,E}^{\,n}(+)}{\left(R_E - h_b\right)\cos(L_b)}\right]$$